Published by Institute of Physics Publishing for SISSA/ISAS
Received: *October 30, 2003*
Accepted: *November 10, 2003*

Master loop equations, free energy and correlations for the chain of matrices

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Abstract: The loop equations for a chain of hermitian random matrices are computed explicitely, including the $1/N^2$ corrections. To leading order, the master loop equation reduces to an algebraic equation, whose solution can be written in terms of geometric properties of the underlying algebraic curve. In particular we compute the free energy, the resolvents, the 2-loop functions and some mixed one loop functions. We also initiate the calculation of the $1/N^2$ expansion.

Keywords: Models of Quantum Gravity, Matrix Models.

*Work partialy supported by the EC IHP Network: "Discrete geometries: from solid state physics to quantum gravity", HPRN-CT-1999-000161.

Contents

1. Introduction

The multimatrix model, or chain of matrices is defined by the probability weight for $\mathcal{N}+1$ hermitean matrices of size $N\times N$:

$$d\mu(M_0,\dots,M_{\mathcal{N}}) \;=\; \frac{1}{Z}dM_0\dots dM_{\mathcal{N}}\mathrm{e}^{-\frac{N}{T}\,\mathrm{tr}[\sum_{k=0}^{\mathcal{N}}V_k(M_k)-\sum_{k=1}^{\mathcal{N}}M_{k-1}M_k]} \tag{1.1}$$

$$Z := \left(\frac{N}{2\pi}\right)^{-\frac{1}{2}(\mathcal{N}+1)N^2}\mathrm{e}^{-\frac{N^2}{T^2}F}$$
$$:= \int dM_0\dots dM_{\mathcal{N}}\mathrm{e}^{-\frac{N}{T}\,\mathrm{tr}[\sum_{k=0}^{\mathcal{N}}V_k(M_k)-\sum_{k=1}^{\mathcal{N}}M_{k-1}M_k]} \tag{1.2}$$

where M_k $(k=0,\dots,\mathcal{N})$ are $N\times N$ hermitean matrices, and dM_k is the product of Lebesgue measures of all real components of M_k. Z is the partition function, F is the free energy. T is the temperature, it can be chosen equal to 1 in most of the paper, except when one is interested in derivatives with respect to T.

The potentials V_k are polynomials of degree $d_k + 1$:

$$V_k(x) = g_{k,0} + \sum_{j=1}^{d_k+1} \frac{g_{k,j}}{j} x^j \,. \tag{1.3}$$

In order for Z to exist, we have to assume that all V_k's are bounded from below on the real axis. However, that constraint can be relaxed by studying "normal matrices" instead of hermitean matrices,[1] or by considering Z as a formal power series in the coefficients of the potentials. For simplicity, we assume here that the V_k's are real.[2]

Each M_k lies in the potential well V_k and is linearly coupled to its neighboors M_{k-1} and M_{k+1}. M_0 and $M_{\mathcal{N}}$ have only one neighboor, and thus the chain is open. So far, the problem of the closed chain has remained unsolved.

The multimatrix model is a generalization of the 2-matrix model. It was often considered in the context of 2-dimensional quantum gravity and string theory. Its critical points are known to represent the minimal conformal field theories, characterized by a pair of integers (p, q). It is known that one can get a (p, q) critical point with a multimatrix model where $\mathcal{N} = q - 2$. The necessity of studying multimatrix models can be understood from the fact that the one matrix models contains only the critical models with $q \leq 2$.

Recent progress have been made in the understanding of the two-matrix model [3, 4, 9, 10, 18, 19], and it has been often noticed that the chain of matrices presents many similarities with the 2-matrix model [7].

The loop equations of the 2-matrix model have been known for a long time [12, 22] and have been concisely written in [10] including the $O(1/N^2)$ terms. The loop equations for the chain of matrices have been written in an appendix of [13] and in a draft of [15] without the $O(1/N^2)$ terms, and without any proof, and also with misprints. The present paper is mainly a translation from french to english of the draft [15], with proofs, and additional results. It thus fills a gap in the appendix of [13]. In particular, the $1/N^2$ terms are included, opening the way to finding next to leading order corrections as in [10]. The leading order solution is also presented in an an updated algebraic geometry language.

The paper is organized as follows:

1. introduction;

2. definitions of the loop functions;

3. the main result and its derivation: the master loop equation. Readers less interested in technical details can skip the derivation;

4. to leading order, the master loop equation is an algebraic equation. We discuss its solution;

5. computation of the free energy in the large N limit, along the lines drawn by [3];

[1] I.e. the eigenvalues are located on complex paths instead of the real axis. For a potential of degree $d_k + 1$, there are d_k homologicaly non equivalent complex paths on which V_k is bounded from below [2].

[2] Most of the results presented here remain valid for complex $V_k's$. The asumption of real potential merely allows to have a more concise derivation of loop equations.

6. computation of other resolvents and correlators in the large-N limit;
7. next to leading order, along the lines drawn by [10, 11];
8. examples (2 and 3-matrix models, gaussian case, one-cut case).

The most important results of this paper are the master loop equation eq. (3.1), the free energy eq. (5.2), the two-loop function eq. (6.5), the mixed correlators eq. (6.8), eq. (6.25) and eq. (6.15).

2. Definitions

2.1 Moments

We define the moments:

$$T_{n_0,n_1,\dots,n_{\mathcal{N}}} := \frac{1}{2N}\left\langle \operatorname{tr} M_0^{n_0} M_1^{n_1}\dots M_{\mathcal{N}}^{n_{\mathcal{N}}}\right\rangle + \frac{1}{2N}\left\langle \operatorname{tr} M_{\mathcal{N}}^{n_{\mathcal{N}}}\dots M_1^{n_1} M_0^{n_0}\right\rangle \tag{2.1}$$

where $\langle\rangle$ means averaging with the probability measure of eq. (1.1). If all the potentials V_k's are bounded from below, it is clear that the moments are well defined convergent integrals. If on the contrary the V_k's are not all bounded from below, then the moments are only formaly defined through their expansions in the coupling constants.

2.2 The complete one loop-function

We define the complete one loop-function as follows:

$$W(z_0,\dots,z_{\mathcal{N}}) := \sum_{n_0,\dots,n_{\mathcal{N}}=0}^{\infty} \frac{T_{n_0,n_1,\dots,n_{\mathcal{N}}}}{z_0^{n_0+1} z_1^{n_1+1}\dots z_{\mathcal{N}}^{n_{\mathcal{N}}+1}} \tag{2.2}$$

which can also be written

$$\begin{aligned} W(z_0,\dots,z_{\mathcal{N}}) = & \frac{1}{2N}\left\langle \operatorname{tr}\frac{1}{z_0-M_0}\frac{1}{z_1-M_1}\cdots\frac{1}{z_{\mathcal{N}}-M_{\mathcal{N}}}\right\rangle + \\ & +\frac{1}{2N}\left\langle \operatorname{tr}\frac{1}{z_{\mathcal{N}}-M_{\mathcal{N}}}\cdots\frac{1}{z_1-M_1}\frac{1}{z_0-M_0}\right\rangle . \end{aligned} \tag{2.3}$$

Notice that it is only a formal definition (even when all the potentials are bounded from below), it makes sense only through its power expansion in the vicinity of all $z_k\to\infty$. That function is merely a convenient concise notation for dealing with all the moments at once.

2.3 Uncomplete one loop-functions

We also define "uncomplete" loop-functions. For any integer k between zero and $\mathcal{N}+1$, and for any ordered subset of $[0,\mathcal{N}]$ $I=\{i_1,\dots,i_k\}$ $(0\le i_1<i_2<\dots<i_k\le\mathcal{N})$, we define:

$$W_{i_1,\dots,i_k}(z_{i_1},\dots,z_{i_k}) := \frac{1}{2N}\left\langle \operatorname{tr}\frac{1}{z_{i_1}-M_{i_1}}\frac{1}{z_{i_2}-M_{i_2}}\cdots\frac{1}{z_{i_k}-M_{i_k}}\right\rangle +$$

$$
\begin{aligned}
&+\frac{1}{2N}\left\langle \operatorname{tr}\frac{1}{z_{i_k}-M_{i_k}}\cdots\frac{1}{z_{i_2}-M_{i_2}}\frac{1}{z_{i_1}-M_{i_1}}\right\rangle \\
=\;& (-1)^{\mathcal{N}+1-k}\operatorname{Res}\prod_{j\notin I} dz_j\, W(z_0,\dots,z_{\mathcal{N}}) \\
=\;& \operatorname*{Pol}_{z_j,\, j\notin I} W(z_0,\dots,z_{\mathcal{N}})\prod_{j\notin I} z_j
\end{aligned}
\tag{2.4}
$$

where Pol means the polynomial part in the vicinity of ∞.

2.4 The resolvents

As a special case of eq. (2.4), the resolvent of the k^{th} matrix of the chain is defined by:

$$
W_k(z_k) := \frac{1}{N}\left\langle \operatorname{tr}\frac{1}{z_k-M_k}\right\rangle \tag{2.5}
$$

In particular, the resolvent of the first matrix of the chain will play a prominent role:

$$
W_0(z_0) := \frac{1}{N}\left\langle \operatorname{tr}\frac{1}{z_0-M_0}\right\rangle . \tag{2.6}
$$

The master loop equation derived in section 3.1 is an equation for $W_0(z_0)$ as a function of z_0, and in section 4, we determine the large-N limit of $W_0(z_0)$. In section 6.1, we derive the large-N limits of the other resolvents W_k.

Instead of $W_0(z_0)$, it appears more convenient to consider the following function:

$$
Z_1(z_0) := V_0'(z_0) - TW_0(z_0) \tag{2.7}
$$

and more generaly we define:

$$
\begin{cases}
Z_{-1}(z_0) := TW_0(z_0) \\
Z_0(z_0) := z_0 \\
Z_{k+1}(z_0) := V_k'(Z_k(z_0)) - Z_{k-1}(z_0) \quad 1\leq k\leq \mathcal{N}
\end{cases}
\tag{2.8}
$$

For short, we write:

$$
Z_k := Z_k(z_0)\,. \tag{2.9}
$$

Notice that all Z_k's are polynomials in both z_0 and $W_0(z_0)$, and therefore, any polynomial of the Z_k's is a polynomial in z_0 and $W_0(z_0)$.

2.5 Polynomial loop functions P and U

We define the functions $f_{k,l}(z_k,\dots,z_l)$:

$$
\begin{cases}
f_{k,l} := 0 \quad \text{if } l<k-1 \\
f_{k,k-1} := 1 \\
f_{k,k}(z_k) := V_k'(z_k) \\
f_{k,l+1}(z_k,\dots,z_{l+1}) := V_{l+1}'(z_{l+1})\, f_{k,l}(z_k,\dots,z_l) - z_l z_{l+1} f_{k,l-1}(z_k,\dots,z_{l-1})\,.
\end{cases}
\tag{2.10}
$$

Then we define for $k \leq l$:

$$P_{k,l}(z_0,\dots,z_{\mathcal{N}}) := \mathop{\mathrm{Pol}}_{z_k,\dots,z_l} f_{k,l}(z_k,\dots,z_l) W(z_0,\dots,z_{\mathcal{N}}) \tag{2.11}$$

where Pol means the polynomial part in the vicinity of ∞. $P_{k,l}$ depends on $\mathcal{N}+1$ variables, and is polynomial in the variables $z_k,\dots,z_l$.

By definition of $f_{k,l}$ we have:

$$P_{k,l}(z_0,\dots,z_{\mathcal{N}}) = \mathop{\mathrm{Pol}}_{z_l} V_l'(z_l) P_{k,l-1}(z_0,\dots,z_{\mathcal{N}}) - \mathop{\mathrm{Pol}}_{z_l,z_{l-1}} z_{l-1}\, z_l\, P_{k,l-2}(z_0,\dots,z_{\mathcal{N}})\,. \tag{2.12}$$

In particular, we define:

$$P(z_0,\dots,z_{\mathcal{N}}) := P_{0,\mathcal{N}}(z_0,\dots,z_{\mathcal{N}}) \tag{2.13}$$

which is polynomial in all the variables, and

$$U(z_0,\dots,z_{\mathcal{N}}) := P_{1,\mathcal{N}}(z_0,\dots,z_{\mathcal{N}}) \tag{2.14}$$

which is polynomial in all the variables but z_0.

We also define:

$$E(z_0,\dots,z_{\mathcal{N}}) := (V_0'(z_0) - z_1)(V_{\mathcal{N}}'(z_{\mathcal{N}}) - z_{\mathcal{N}-1}) - TP(z_0,\dots,z_{\mathcal{N}}) \tag{2.15}$$

which is also a polynomial in all variables. These polynomials will play an important role in the master loop equation.

2.6 2-loop functions

We define the loop insertion operator as in [1]:

$$\frac{\partial}{\partial V_l(z)} := -\frac{1}{z}\frac{\partial}{\partial g_{l,0}} - \sum_{k=1}^{\infty} \frac{k}{z^{k+1}}\frac{\partial}{\partial g_{l,k}} \tag{2.16}$$

where $V_l(z) = g_{l,0} + \sum_{k\geq 1} \frac{g_{l,k}}{k} z^k$, and the derivatives are taken at $g_{l,k} = 0$ for $k > d_l + 1$. It is a formal definition, which makes sense only order by order in its large z expansion.

It is such that (for $|z| > |z'|$):

$$\frac{\partial V_k(z')}{\partial V_l(z)} = -\delta_{k,l}\frac{1}{z-z'}\,, \qquad \frac{\partial V_k'(z')}{\partial V_l(z)} = -\delta_{k,l}\frac{1}{(z-z')^2}\,. \tag{2.17}$$

When applied to eq. (1.2), it produces a resolvent, i.e. the expectation of a trace:

$$\frac{\partial F}{\partial V_l(z)} = -TW_l(z) = -\frac{T}{N}\left\langle \mathrm{tr}\,\frac{1}{z-M_l}\right\rangle \tag{2.18}$$

and more generaly, the action of $\partial/\partial V_l(z)$ on an expectation value inserts a new trace.

In particular, we define the following two loop functions (i.e. two-traces):

$$\begin{aligned}
W_{;l}(z_0,\ldots,z_{\mathcal{N}};z) &:= T\frac{\partial}{\partial V_l(z)}W(z_0,\ldots,z_{\mathcal{N}}) \\
&= \frac{1}{2}\left\langle \operatorname{tr}\frac{1}{z_0-M_0}\frac{1}{z_1-M_1}\cdots\frac{1}{z_{\mathcal{N}}-M_{\mathcal{N}}}\operatorname{tr}\frac{1}{z-M_l}\right\rangle + \\
&\quad +\frac{1}{2}\left\langle \operatorname{tr}\frac{1}{z-M_l}\operatorname{tr}\frac{1}{z_{\mathcal{N}}-M_{\mathcal{N}}}\cdots\frac{1}{z_1-M_1}\frac{1}{z_0-M_0}\right\rangle - \\
&\quad -N^2 W_l(z)W(z_0,\ldots,z_{\mathcal{N}}) \qquad (2.19)\\
W_{i_1,\ldots,i_k;l}(z_{i_1},\ldots,z_{i_k};z) &:= T\frac{\partial}{\partial V_l(z)}W_{i_1,\ldots,i_k}(z_{i_1},\ldots,z_{i_k}) \\
&= (-1)^{\mathcal{N}+1-k}\operatorname{Res}\prod_{j\notin I} dz_j\, W_{;l}(z_0,\ldots,z_{\mathcal{N}};z) \qquad (2.20)\\
P_{k,l;j}(z_0,\ldots,z_{\mathcal{N}};z) &:= \operatorname*{Pol}_{z_k,\ldots,z_l} f_{k,l}(z_k,\ldots,z_l)W_{;j}(z_0,\ldots,z_{\mathcal{N}};z) \qquad (2.21)\\
P_{;j}(z_0,\ldots,z_{\mathcal{N}};z) &:= P_{0,\mathcal{N};j}(z_0,\ldots,z_{\mathcal{N}};z) \qquad (2.22)\\
U_{;j}(z_0,\ldots,z_{\mathcal{N}};z) &:= P_{1,\mathcal{N};j}(z_0,\ldots,z_{\mathcal{N}};z)\,. \qquad (2.23)
\end{aligned}$$

In particular, the following two-loop functions will play an important role:

$$\begin{aligned}
W_{k;l}(z';z) &= T\frac{\partial W_k(z')}{\partial V_l(z)} = -\frac{\partial^2 F}{\partial V_k(z')\partial V_l(z)} = T\frac{\partial W_l(z)}{\partial V_k(z')} = W_{l;k}(z;z') \\
&= \left\langle \operatorname{tr}\frac{1}{z-M_l}\operatorname{tr}\frac{1}{z'-M_k}\right\rangle_{\mathrm{c}} \qquad (2.24)
\end{aligned}$$

where the subscript c means connected part $< AB >_{\mathrm{c}} := < AB > - < A >< B >$.

3. Loop equations

3.1 Main result: the master loop equation

We prove below that the master loop equation (named after [22]) can be written:

$$\boxed{E(Z_0,Z_1,\ldots,Z_{\mathcal{N}}) = \frac{T^2}{N^2}U_{;0}(Z_0,Z_1,\ldots,Z_{\mathcal{N}};Z_0)} \qquad (3.1)$$

where the functions $Z_k(z_0)$ have been defined in eq. (2.8). Examples for $\mathcal{N}=1$ (the 2-matrix model) and $\mathcal{N}=2$ (the 3-matrix model) are explicited in section 8.

As an intermediate step in proving eq. (3.1), we need to prove the following formula for all $1\leq k\leq\mathcal{N}$:

$$\boxed{\begin{aligned}
(z_k-Z_k)W_{0,k,\ldots,\mathcal{N}}(z_0,z_k,\ldots,z_{\mathcal{N}}) = {} & W_{0,k+1,\ldots,\mathcal{N}}(z_0,z_{k+1},\ldots,z_{\mathcal{N}}) \\
& -P_{0,k-1}(z_0,Z_1,\ldots,Z_{k-1},z_k,\ldots,z_{\mathcal{N}}) \\
& -\frac{T}{N^2}P_{1,k-1;0}(z_0,Z_1,\ldots,Z_{k-1},z_k,\ldots,z_{\mathcal{N}};z_0)
\end{aligned}} \qquad (3.2)$$

3.2 Derivation of the master loop equation

- Proof of eq. (3.2) for $k=1$:

  The invariance of the matrix integral under the infinitesimal change of variable (see [14, 8, 10], and pay attention to the non-commutativity of matrices, and to the hermiticity of the change of variables):

$$\begin{aligned}\delta M_0 = & \frac{T}{2}\frac{1}{z_1-M_1}\frac{1}{z_2-M_2}\cdots\frac{1}{z_{\mathcal{N}}-M_{\mathcal{N}}}\frac{1}{z_0-M_0}+ \\ & +\frac{T}{2}\frac{1}{z_0-M_0}\frac{1}{z_{\mathcal{N}}-M_{\mathcal{N}}}\cdots\frac{1}{z_2-M_2}\frac{1}{z_1-M_1}\end{aligned} \tag{3.3}$$

  implies

$$\begin{aligned}& TW_0(z_0)W(z_0,z_1,\dots,z_{\mathcal{N}})+\frac{T}{N^2}W_{;0}(z_0,z_1,\dots,z_{\mathcal{N}};z_0)= \\ & \quad =V_0'(z_0)W(z_0,z_1,\dots,z_{\mathcal{N}})-P_{0,0}(z_0,z_1,\dots,z_{\mathcal{N}})- \\ & \quad\quad -z_1W_{0,1,\dots,\mathcal{N}}(z_0,z_1,\dots,z_{\mathcal{N}})+W_{0,2,\dots,\mathcal{N}}(z_0,z_2,\dots,z_{\mathcal{N}})\end{aligned} \tag{3.4}$$

  The l.h.s. comes from the jacobian of the change of variable, and the r.h.s. comes from the variation of the action. The first two terms of the r.h.s. come from $\delta\,\mathrm{tr}\,V_0(M_0)=\mathrm{tr}\,V_0'(M_0)\delta M_0$, and use of eq. (A.5). The last two terms of the r.h.s. come from $\delta\,\mathrm{tr}\,M_0M_1=\mathrm{tr}\,M_1\delta M_0$, and use of eq. (A.6).

  Using $Z_1=V_0'(z_0)-TW_0(z_0)$, this can be rewritten:

$$\begin{aligned}(z_1-Z_1)W_{0,1,\dots,\mathcal{N}}(z_0,z_1,\dots,z_{\mathcal{N}}) = & W_{0,2,\dots,\mathcal{N}}(z_0,z_2,\dots,z_{\mathcal{N}})- \\ & -P_{0,0}(z_0,z_1,\dots,z_{\mathcal{N}})- \\ & -\frac{T}{N^2}W_{;0}(z_0,z_1,\dots,z_{\mathcal{N}};z_0)\,.\end{aligned} \tag{3.5}$$

  Therefore we have derived eq. (3.2) in the case $k=1$.

- Proof of eq. (3.2) for $k=2$:

  Notice that eq. (3.5) implies:

$$\begin{aligned}-\mathrm{Res}\,dz_1\,W(z_0,z_1,z_2,\dots,z_{\mathcal{N}})V_1'(z_1) = & V_1'(Z_1)W_{0,2,\dots,\mathcal{N}}(z_0,z_2,\dots,z_{\mathcal{N}})- \\ & -P_{0,1}(z_0,Z_1,z_2,\dots,z_{\mathcal{N}})- \\ & -W_{2,\dots,\mathcal{N}}(z_2,\dots,z_{\mathcal{N}})- \\ & -\frac{T}{N^2}P_{1,1;0}(z_0,Z_1,z_2,\dots,z_{\mathcal{N}};z_0)\,.\end{aligned} \tag{3.6}$$

  The change of variable:

$$\delta M_1=\frac{1}{2}\frac{1}{z_2-M_2}\cdots\frac{1}{z_{\mathcal{N}}-M_{\mathcal{N}}}\frac{1}{z_0-M_0}+\frac{1}{2}\frac{1}{z_0-M_0}\frac{1}{z_{\mathcal{N}}-M_{\mathcal{N}}}\cdots\frac{1}{z_2-M_2} \tag{3.7}$$

  gives (since δM_1 is independant of M_1 there is no jacobian, the l.h.s. is zero, and the r.h.s. is the variation of $\mathrm{tr}\,V_1'(M_1)-M_0M_1-M_1M_2$):

$$0 = -\mathrm{Res}\,dz_1\,W(z_0,z_1,z_2,\dots,z_{\mathcal{N}})V_1'(z_1)-$$

$$-z_0 W_{0,2,\dots,\mathcal{N}}(z_0,z_2,\dots,z_{\mathcal{N}}) + W_{2,\dots,\mathcal{N}}(z_2,\dots,z_{\mathcal{N}}) -$$
$$-z_2 W_{0,2,\dots,\mathcal{N}}(z_0,z_2,\dots,z_{\mathcal{N}}) + W_{0,3,\dots,\mathcal{N}}(z_0,z_3,\dots,z_{\mathcal{N}}) \tag{3.8}$$

i.e., using eq. (3.6) and $V_1'(Z_1) = z_0 + Z_2$:

$$\begin{aligned}(z_2 - Z_2) W_{0,2,\dots,\mathcal{N}}(z_0,z_2,\dots,z_{\mathcal{N}}) \;=\; & W_{0,3,\dots,\mathcal{N}}(z_0,z_3,\dots,z_{\mathcal{N}}) - \\ & -P_{0,1}(z_0,Z_1,z_2,\dots,z_{\mathcal{N}}) - \\ & -\frac{T}{N^2} P_{1,1;0}(z_0,Z_1,z_2,\dots,z_{\mathcal{N}};z_0)\,. \end{aligned} \tag{3.9}$$

Therefore we have derived eq. (3.2) in the case $k = 2$.

- Proof of eq. (3.2) by recursion on k:

  We have already proved eq. (3.2) for $k = 1$ and $k = 2$. Now assume that eq. (3.2) holds for $k-1$ and k, we are going to prove it for $k+1$.

  Note that eq. (3.2) for $k-1$ implies (multiply by z_{k-1} and take the residue at $z_{k-1} \to \infty$ and $z_k \to \infty$):

$$\begin{aligned}&\mathrm{Res}\, dz_{k-1}\, \mathrm{Res}\, dz_k\;\; z_{k-1} W_{0,k-1,k,\dots,\mathcal{N}}(z_0,z_{k-1},z_k,z_{k+1},\dots,z_{\mathcal{N}}) = \\ &\quad = Z_{k-1} W_{0,k+1,\dots,\mathcal{N}}(z_0,z_{k+1},\dots,z_{\mathcal{N}}) - \\ &\quad\; - \mathop{\mathrm{Pol}}_{z_{k-1},z_k}\; z_{k-1} z_k P_{0,k-2}(z_0,\dots,z_{\mathcal{N}})|_{z_0=Z_0,\dots,z_k=Z_k} - \\ &\quad\; - \frac{T}{N^2} \mathop{\mathrm{Pol}}_{z_{k-1},z_k}\; z_k z_{k-1} P_{1,k-2;0}(z_0,\dots,z_{\mathcal{N}};z_0)|_{z_0=Z_0,\dots,z_k=Z_k}\end{aligned} \tag{3.10}$$

  and eq. (3.2) for k implies (multiply by $V_k'(z_k)$ and take the residue at $z_k \to \infty$):

$$\begin{aligned}&-\mathrm{Res}\, dz_k\;\, V_k'(z_k)\, W_{0,k,k+1,\dots,\mathcal{N}}(z_0,z_k,z_{k+1},\dots,z_{\mathcal{N}}) = \\ &\quad = V_k'(Z_k) W_{0,k+1,\dots,\mathcal{N}}(z_0,z_{k+1},\dots,z_{\mathcal{N}}) - \\ &\quad\; - \mathop{\mathrm{Pol}}_{z_k}\; V_k'(z_k) P_{0,k-1}(z_0,\dots,z_{\mathcal{N}})\big|_{z_0=Z_0,\dots,z_k=Z_k} - \\ &\quad\; - \frac{T}{N^2} \mathop{\mathrm{Pol}}_{z_k}\; V_k'(z_k) P_{1,k-1;0}(z_0,\dots,z_{\mathcal{N}};z_0)\big|_{z_0=Z_0,\dots,z_k=Z_k}\,.\end{aligned} \tag{3.11}$$

  Then consider the change of variable:

$$\delta M_k = \frac{1}{2}\frac{1}{z_{k+1}-M_{k+1}}\cdots\frac{1}{z_{\mathcal{N}}-M_{\mathcal{N}}}\frac{1}{z_0-M_0} + \frac{1}{2}\frac{1}{z_0-M_0}\frac{1}{z_{\mathcal{N}}-M_{\mathcal{N}}}\cdots\frac{1}{z_{k+1}-M_{k+1}} \tag{3.12}$$

  it gives

$$\begin{aligned}0 = & -\mathrm{Res}\, dz_k\, V_k'(z_k)\, W_{0,k,k+1,\dots,\mathcal{N}}(z_0,z_k,z_{k+1},\dots,z_{\mathcal{N}}) - \\ & -\mathrm{Res}\, dz_{k-1}\,\mathrm{Res}\, dz_k\;\, z_{k-1} W_{0,k-1,\dots,\mathcal{N}}(z_0,z_{k-1},\dots,z_{\mathcal{N}}) - \\ & -z_{k+1} W_{0,k+1,\dots,\mathcal{N}}(z_0,z_{k+1},\dots,z_{\mathcal{N}}) + W_{0,k+2,\dots,\mathcal{N}}(z_0,z_{k+2},\dots,z_{\mathcal{N}})\end{aligned} \tag{3.13}$$

  using eq. (3.10) and eq. (3.11), as well as $V_k'(Z_k) = Z_{k-1} + Z_{k+1}$, we get eq. (3.2) for $k+1$.

- eq. (3.2) for $k = \mathcal{N}$:

  In particular, the previous recurrence derivation shows that eq. (3.2) for $k = \mathcal{N}$ reads:

$$\begin{aligned}(z_{\mathcal{N}} - Z_{\mathcal{N}})W_{0,\mathcal{N}}(z_0, z_{\mathcal{N}}) = &\, W_0(z_0) - P_{0,\mathcal{N}-1}(z_0, Z_1, \ldots, Z_{\mathcal{N}-1}, z_{\mathcal{N}}) - \\ & - \frac{T}{N^2} P_{1,\mathcal{N}-1;0}(z_0, Z_1, \ldots, Z_{\mathcal{N}-1}, z_{\mathcal{N}}; z_0)\,. \end{aligned} \tag{3.14}$$

- Proof of eq. (3.1):

  In particular, from eq. (3.2) for $k = \mathcal{N} - 1$ we derive:

$$\begin{aligned}&\mathrm{Res}\, dz_{\mathcal{N}-1}\, \mathrm{Res}\, dz_{\mathcal{N}}\, z_{\mathcal{N}-1} W_{0,z_{\mathcal{N}-1},z_{\mathcal{N}}}(z_0, z_{\mathcal{N}-1}, z_{\mathcal{N}}) = \\ &\quad = Z_{\mathcal{N}-1} W_0(z_0) - - \mathop{\mathrm{Pol}}_{z_{\mathcal{N}-1},z_{\mathcal{N}}} z_{\mathcal{N}-1} z_{\mathcal{N}} P_{0,\mathcal{N}-2}(z_0, \ldots, z_{\mathcal{N}})\big|_{z_0=Z_0,\ldots,z_{\mathcal{N}}=Z_{\mathcal{N}}} \\ &\quad - \frac{T}{N^2} \mathop{\mathrm{Pol}}_{z_{\mathcal{N}-1},z_{\mathcal{N}}} z_{\mathcal{N}-1} z_{\mathcal{N}} P_{1,\mathcal{N}-1;0}(z_0, \ldots, z_{\mathcal{N}}; z_0)\big|_{z_0=Z_0,\ldots,z_{\mathcal{N}}=Z_{\mathcal{N}}} \end{aligned} \tag{3.15}$$

  and from eq. (3.14), we derive:

$$\begin{aligned}&- \mathrm{Res}\, dz_{\mathcal{N}}\, V'_{\mathcal{N}}(z_{\mathcal{N}})\, W_{0,\mathcal{N}}(z_0, z_{\mathcal{N}}) = \\ &\quad = V'_{\mathcal{N}}(Z_{\mathcal{N}}) W_0(z_0) - \\ &\quad - \mathop{\mathrm{Pol}}_{z_{\mathcal{N}}} V'_{\mathcal{N}}(z_{\mathcal{N}}) P_{0,\mathcal{N}-1}(z_0, \ldots, z_{\mathcal{N}})\big|_{z_0=Z_0,\ldots,z_{\mathcal{N}}=Z_{\mathcal{N}}} - \\ &\quad - \frac{T}{N^2} \mathop{\mathrm{Pol}}_{z_{\mathcal{N}}} V'_{\mathcal{N}}(z_{\mathcal{N}}) P_{1,\mathcal{N}-1;0}(z_0, \ldots, z_{\mathcal{N}}; z_0)\big|_{z_0=Z_0,\ldots,z_{\mathcal{N}}=Z_{\mathcal{N}}}\,. \end{aligned} \tag{3.16}$$

  Then, the change of variables

$$\delta M_{\mathcal{N}} = \frac{1}{z_0 - M_0} \tag{3.17}$$

  gives

$$\begin{aligned}0 = &- \mathrm{Res}\, dz_{\mathcal{N}}\, V'_{\mathcal{N}}(z_{\mathcal{N}})\, W_{0,\mathcal{N}}(z_0, z_{\mathcal{N}}) - \\ &- \mathrm{Res}\, dz_{\mathcal{N}-1}\, \mathrm{Res}\, dz_{\mathcal{N}}\, z_{\mathcal{N}-1} W_{0,z_{\mathcal{N}-1},z_{\mathcal{N}}}(z_0, z_{\mathcal{N}-1}, z_{\mathcal{N}}) \end{aligned} \tag{3.18}$$

  i.e., using eq. (3.15) and eq. (3.16):

$$\begin{aligned}0 = &\,(V'_{\mathcal{N}}(Z_{\mathcal{N}}) - Z_{\mathcal{N}-1}) W_0(z_0) - P_{0,\mathcal{N}}(Z_0, \ldots, Z_{\mathcal{N}}) - \\ & - \frac{T}{N^2} P_{1,\mathcal{N};0}(Z_0, \ldots, Z_{\mathcal{N}}; z_0)\,. \end{aligned} \tag{3.19}$$

  Recalling that $Z_1 = V'_0(z_0) - T W_0(z_0)$, we get:

$$\begin{aligned}&(V'_0(Z_0) - Z_1)(V'_{\mathcal{N}}(Z_{\mathcal{N}}) - Z_{\mathcal{N}-1}) - T P_{0,\mathcal{N}}(Z_0, Z_1, \ldots, Z_{\mathcal{N}}) = \\ &\quad = \frac{T^2}{N^2} P_{1,\mathcal{N};0}(Z_0, Z_1, \ldots, Z_{\mathcal{N}}; Z_0) \end{aligned} \tag{3.20}$$

  which ends the proof of eq. (3.1).

4. Large N leading order, algebraic equation

Throughough this section and sections 5 and 6, we abusively denote with the same name, the loop functions and their large-N limits.

To leading order, the master loop equation reduces to an algebraic equation for W_0 as a function of z_0:

$$E(z_0, Z_1, \dots, Z_{\mathcal{N}}) = 0 \tag{4.1}$$

where $Z_1 = V_0'(z_0) - TW_0(z_0)$ and $Z_{k+1} = V_k'(Z_k) - Z_{k-1}$.

E is a polynomial of given degrees in each variable, and with known leading coefficients. The problem is that most of the subleading coefficients of $E(z_0, z_1, \dots, z_{\mathcal{N}})$ are not determined by the loop equations. They are determined by additional hypothesis, which will be explained in section 4.5. Prior to that, we need to study the geometry of that algebraic equation.

The complex curve W_0 (equivalently Z_1) as a function of z_0 is a one–dimensional submanifold of $\mathbf{C}^{\mathcal{N}+1}$, which is the intersection of $\mathcal{N}$ dimension $\mathcal{N}$ algebraic submanifolds. The curve W_0 as a function of z_0 is thus a Riemann surface $\mathcal{E}$.

Instead of viewing the Z_k's as (mutivalued) functions of z_0, it is more appropriate to view the Z_k's as (monovalued) complex functions over $\mathcal{E}$:

$$p \to z_k(p) \tag{4.2}$$

such that for all $p \in \mathcal{E}$:

$$Z_k(z_0(p)) = z_k(p)\,. \tag{4.3}$$

Since the function $z_0(p)$ is not injective, i.e. the point p such that $z_0(p) = z_0$ is not unique, the Z_k's are multivalued functions of z_0. Similarly, one could consider any z_k as a multivalued function of any z_l.

Instead of dealing with multivalued functions, we slice $\mathcal{E}$ into domains called sheets, such that in each domain the functions we are considering are bijections. The z_0-sheets are thus domains in which the function $z_0(p)$ is injective. More generaly, the z_k-sheets are domains in which the function $z_k(p)$ is injective.

Let us study the z_0-sheets first. For that purpose, we define:

$$\begin{cases} r_{-1} := -1 \\ r_0 \;\; := 1 \\ r_k \;\; := d_0 d_1 \dots d_{k-1}\,, \quad k = 1, \dots, \mathcal{N}+1 \end{cases}$$
$$\begin{cases} s_{\mathcal{N}+1} := -1 \\ s_{\mathcal{N}} \;\;\; := 1 \\ s_k \;\;\;\; := d_{k+1} \dots d_{\mathcal{N}-1} d_{\mathcal{N}}\,, \quad k = -1, \dots, \mathcal{N}-1\,. \end{cases} \tag{4.4}$$

4.1 z_0-sheets

The equation

$$E(z_0, Z_1, \dots, Z_{\mathcal{N}}) = 0 \tag{4.5}$$

has degree $1 + d_1 d_2 \ldots d_{\mathcal{N}}$ in W_0 and degree $d_0 d_1 \ldots d_{\mathcal{N}}$ in z_0. Therefore, $W_0(z_0)$ (or equivalently Z_1) is a multivalued function of z_0 which takes $1+d_1 d_2 \ldots d_{\mathcal{N}}$ values: in other words, there are $1 + d_1 d_2 \ldots d_{\mathcal{N}}$ z_0-sheets. We identify these sheets by looking at the asymptotics of $Z_{\mathcal{N}}$ when $z_0 \to \infty$.

- The physical sheet

  From eq. (2.6), there must exist at least one solution of the algebraic equation such that:
  $$W_0(z_0) \underset{z_0\to\infty}{\sim} \frac{1}{z_0} + O\left(\frac{1}{z_0^2}\right) \tag{4.6}$$
  which implies that:
  $$z_k = O(z_0^{r_k}) \tag{4.7}$$
  and in particular
  $$z_{\mathcal{N}} = O(z_0^{d_0 d_1 \ldots d_{\mathcal{N}-1}})\,. \tag{4.8}$$
  The z_0-sheet in which these asymptotics hold is called the z_0-physical sheet.

- Other sheets

  Notice that the equation for $W_{\mathcal{N}}(z_{\mathcal{N}})$ as a function of $z_{\mathcal{N}}$ is the same algebraic equation. In other words, there is a solution of the algebraic equation which is such that:
  $$z_k = O(z_{\mathcal{N}}^{s_k})\,, \qquad s_k = d_{k+1} \ldots d_{\mathcal{N}-1} d_{\mathcal{N}} \tag{4.9}$$
  i.e.
  $$z_{\mathcal{N}} = O(z_0^{1/d_1 \ldots d_{\mathcal{N}-1} d_{\mathcal{N}}})\,. \qquad z_k = O(z_0^{1/d_1 \ldots d_k})\,. \tag{4.10}$$
  Since the number of s_0^{th} roots of unity is exactly $s_0 = d_1 \ldots d_{\mathcal{N}-1} d_{\mathcal{N}}$, we have all the solutions.

We thus have $r_0 + s_0 = 1 + d_1 \ldots d_{\mathcal{N}-1} d_{\mathcal{N}}$ sheets. In one of them we have $z_{\mathcal{N}} \sim O(z_0^{r_{\mathcal{N}}})$, and in the $d_1 \ldots d_{\mathcal{N}-1} d_{\mathcal{N}}$ others we have $z_0 \sim O(z_{\mathcal{N}}^{s_0})$.

4.2 Algebraic geometry

We now consider the algebraic curve $\mathcal{E}$ in a more geometric language. An abstract point $p \in \mathcal{E}$ can be represented as a couple (z_0, z_1) such that $z_1 = Z_1(z_0)$, or by any other parametrization. For instance it can be described as a point of $\mathbf{C}^{\mathcal{N}}$, at the intersection of $\mathcal{N}$ codimension 1 manifolds.

Algebraic geometry is an active and important part of mathematics, and lots of tools have been invented to describe the geometry of algebraic Riemann surfaces. We refer the reader to [16, 17] for an introduction.

A Riemann surface is locally homomorphic to the complex plane $\mathbf{C}$, that means that small domains of $\mathcal{E}$ can be maped on small domains of $\mathbf{C}$. The map, which is one-to-one and holomorphic in that domain is called a local parameter, let us call it:
$$p \in \mathcal{E} \to x(p) \in \mathbf{C}\,. \tag{4.11}$$

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Any complex valued analytic function on $\mathcal{E}$ can be localy represented by an analytic function of $x(p)$.

For instance, $z_0 = z_0(p)$ is often a good local parameter on $\mathcal{E}$, except when z_0 approaches a singularity. Therefore all functions on $\mathcal{E}$ can be localy written as functions of z_0. They can also be locally written as functions of any z_k.

Notice that the function $z_0(p)$ is not injective, it takes the same value z_0 for different points p, namely the same value of z_0 corresponds to $1+s_0$ points p. Therefore the function $z_1(p)$ which is a well defined monovalued function on $\mathcal{E}$ is a multivalued function of z_0. The z_0-sheets are domains of $\mathcal{E}$ where the function $z_0(p)$ is injective. In particular in each sheet, there is only one point p_∞ where $z_0 \to \infty$.

Let $p_{\infty+}$ be the point in the physical sheet, such that $z_0(p_{\infty+}) = \infty$. In the vicinity of $p_{\infty+}$, we have eq. (4.7):

$$z_k(p) \underset{p\to p_{\infty+}}{\sim} z_0^{r_k}(p) \tag{4.12}$$

i.e. $Z_k(z_0)$ is analytical, which indicates that z_0 is a good local parameter near $p_{\infty+}$.

Now, let $p_{\infty-}$ be the point at ∞ in the $z_\mathcal{N}$-physical sheet, i.e. where $z_\mathcal{N}(p_{\infty-}) = \infty$ and with behavior eq. (4.9):

$$z_k(p) \underset{p\to p_{\infty-}}{\sim} z_\mathcal{N}^{s_k}(p) \tag{4.13}$$

i.e. $Z_k(z_\mathcal{N})$ is analytical, which indicates that $z_\mathcal{N}$ is a good local parameter near $p_{\infty-}$. In the vicinity of $p_{\infty-}$, the function $z_0(p)$ takes the same value z_0, s_0 times, therefore there are s_0 sheets which meet at $p_{\infty-}$. It is clear that z_0 is not a local parameter near $p_{\infty-}$, but $z_\mathcal{N}$ is.

It is also clear that there can be no other point p such that $z_0(p) = \infty$. The algebraic curve $\mathcal{E}$ has only two points at ∞.

Notice that the intermediate z_k's with $0 < k < \mathcal{N}$ are not appropriate local coordinates near $p_{\infty+}$ neither near $p_{\infty-}$ (unless many of the d_k's are equal to 1).

Let us summarize as follows:

- The function $z_k(p)$ has a pole of degree $r_k = d_0 d_1 \dots d_{k-1}$ near $p_{\infty+}$. $z_0(p)$ is a local parameter near $p_{\infty+}$;
- The function $z_k(p)$ has a pole of degree $s_k = d_{k+1} \dots d_{\mathcal{N}-1} d_\mathcal{N}$ near $p_{\infty-}$. $z_\mathcal{N}(p)$ is a local parameter near $p_{\infty-}$;
- The function $z_k(p)$ has no other pole.

Remark. From eq. (4.6), it is easy to prove by recursion that:

$$\operatorname*{Res}_{p_{\infty+}} z_{k-1} dz_k = -T = -\operatorname*{Res}_{p_{\infty+}} z_{k+1} dz_k\,, \qquad \operatorname*{Res}_{p_{\infty-}} z_{k+1} dz_k = -T = -\operatorname*{Res}_{p_{\infty-}} z_{k-1} dz_k\,. \tag{4.14}$$

4.2.1 Genus and cycles

Let g be the genus of $\mathcal{E}$. It can be proved (using the Riemann-Roch theorem, or the method of [19]) that

$$g \le g_{\max} := \prod_{k=0}^{\mathcal{N}} d_k \; - 1\,. \tag{4.15}$$

Notice that the polynomial P has $1 + g_{\max}$ coefficients, and its leading coefficient is fixed, i.e. $g_{\max}$ is the number of coefficients of P not fixed by the loop equations.

Let $\mathcal{A}_i, \mathcal{B}_i (i = 1, \dots, g)$ be a canonical basis of non-trivial cycles on $\mathcal{E}$:

$$\mathcal{A}_i \bigcap \mathcal{B}_j = \delta_{ij} \tag{4.16}$$

the choice of non-trivial cycles is not unique, and we will see below one possible convenient choice.

4.2.2 Endpoints

If z_l is considered as a function of z_k ($k \neq l$), it has singularities (branch points) everytime that:

$$dz_k(p) = 0\,. \tag{4.17}$$

Indeed at such a point we have: $z_l'(z_k) = dz_l/dz_k \to \infty$.

The zeroes of $dz_k(p)$ are called the endpoints, and are noted:

$$e_{k,i}\,, \quad (i = 1, \dots, r_k + s_k + 2g)\,. \tag{4.18}$$

Generically, the zeroes of dz_k are simple zeroes and they are all distinct, which means that z_k behaves as a quadratic function of a local parameter x, while z_l behaves linearly in x. Therefore, $z_l(z_k)$ has a square root branch point near $z_k(e_{k,i})$.

4.2.3 Critical points

It may happen that some endpoints coincide. This is called a critical point. It is not a generic situation, it happens only if the potentials V_k's are fine tuned to some critical potentials. The critical points are relevant for finding the representations of (p, q) minimal conformal models. A (p, q) critical point, is such that there exist $k \neq l$, and a point $e \in \mathcal{E}$ such that: dz_k has a zero of degree $p - 1$ at e and dz_l has a zero of degree $q - 1$ at e. Near e, z_k as a function of z_l behaves with a p/q exponent.

From now on, we assume that the potentials V_k are not critical, i.e. the endpoints are all simple and distinct.

4.2.4 Cuts

The cuts in $\mathcal{E}$ are the contours which border the sheets. Viewed in the z_k-plane they are lines in $\mathbf{C}$ joining two endpoints.

That choice of lines (i.e. the choice of sheets) is largely arbitrary, provided that they indeed go through the endpoints.

There is a canonical choice, which comes from [13], given by the contours where the asymptotics of the biorthogonal polynomials and their Fourier and Hilbert transforms, have discontinuities, i.e. some Stokes lines. That canonical choice is defined as follows: the set of z_k-cuts (contours on $\mathcal{E}$) is the set of points $p \in \mathcal{E}$ such that there exists $p' \neq p$ with $z_k(p') = z_k(p)$ and

$$\Re\left(\int_p^{p'} Z_{k+1} dz_k\right) = 0 \tag{4.19}$$

remark that for all p and p' such that $z_k(p) = z_k(p')$ we have:

$$\int_p^{p'} z_{k+1} dz_k = \int_p^{p'} (V_k'(z_k) - z_{k-1}) dz_k = - \int_p^{p'} z_{k-1} dz_k \tag{4.20}$$

so that the canonical cuts are left unchanged if we reverse the order of the chain.

4.3 Sheet geometry

We are going to describe briefly the sheet geometry, see figure (1) for a better vizualization.

The equation $z_k(p) = z_k$ has $r_k + s_k$ solutions for p, i.e. the curve $\mathcal{E}$ is divided into $r_k + s_k$ sheets. When $z_k \to \infty$, r_k solutions approach $p_{\infty+}$, while s_k approach $p_{\infty-}$. In other words, r_k sheets contain $p_{\infty+}$, while s_k sheets contain $p_{\infty-}$.

Let us denote $\mathcal{C}_k$ the contour which separates the reunion of sheets containing $p_{\infty+}$ from the reunion of sheets containing $p_{\infty-}$, oriented such that it turns around $p_{\infty+}$ in the positive direction. That contour $\mathcal{C}_k$ will play an important role later. Note that $\mathcal{C}_k$ is not necessarily connected (as in our example figure 1).

Let us also denote:

- $p_{+j,k}(z_k)$ $(j = 1, \ldots, r_k)$ all the solutions of $z_k(p) = z_k$ which are on the same side of $\mathcal{C}_k$ than $p_{\infty+}$,

- • $p_{-j,k}(z_k)$ $(j = 1, \ldots, s_k)$ all the solutions of $z_k(p) = z_k$ which are on the same side of $\mathcal{C}_k$ than $p_{\infty-}$.

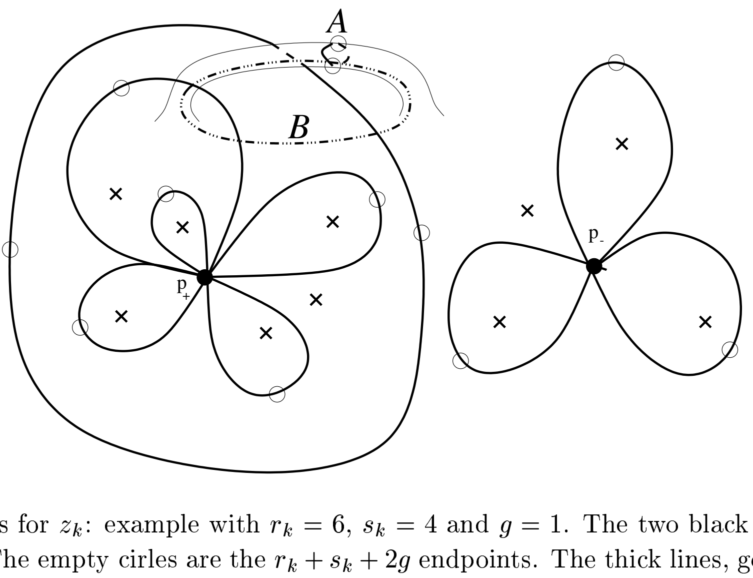

Figure 1: Sheets for z_k: example with $r_k = 6$, $s_k = 4$ and $g = 1$. The two black cirlces represent $p_{\infty+}$ and $p_{\infty-}$. The empty cirles are the $r_k + s_k + 2g$ endpoints. The thick lines, going through the endpoints, are the cuts separating the sheets. Each region separated by lines is a sheet, there are $r_k + s_k$ sheets. The sheets form "flowers" near $p_{\infty+}$ and $p_{\infty-}$ (the petals have an angle $2\pi/r_k$ near $p_{\infty+}$ and $2\pi/s_k$ near $p_{\infty-}$). There are g "handles", with $\mathcal{A}$ and $\mathcal{B}$ cycles: here $\mathcal{A}$ can be chosen as a cut, and $\mathcal{B}$ is represented as a thick dash-dot line. Each cross $\mathbf{x}$ (one in each sheet) represents a point $p \in \mathcal{E}$, such that $z_k(p) = z_k$.

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4.4 Filling fractions

It is well known that the discontinuities of $W_0(z_0)$, i.e. the discontinuities of $Z_{-1}(z_0)$ along the Z_0-cuts in the physical sheet are related to the large-N average density of eigenvalues of the matrix M_0:

$$\rho_0(z_0) = -\frac{1}{2i\pi}(W_0(z_0+i0) - W_0(z_0-i0))\,. \tag{4.21}$$

The support of ρ_0 is the set of cuts of $Z_{-1}(z_0)$, with endpoints the $z_0(e_{0,i})$'s which belong to the physical sheet.

The ratio of the average number of eigenvalues in a given connected component of the support of ρ_0 to the total number N of eigenvalues is a contour integral:

$$\frac{1}{T}\epsilon_i = \int_{z_0(e_{0,i})}^{z_0(e_{0,i+1})} \rho(z_0)dz_0 = -\frac{1}{2i\pi T}\oint z_{-1}dz_0 \tag{4.22}$$

where the contour of integration on $\mathcal{E}$ is one of the z_0–cuts defined in section 4.2.4, oriented in the clockwise direction. ϵ_i is non-zero only if the contour is a non-trivial contour, or if it encloses a pole $p_{\infty+}$ or $p_{\infty-}$.

Therefore, there are at most $g+1$ possible values of ϵ_i, the support of the density has at most $g+1$ connected components.

It is possible to choose the potentials $V_1,\ldots,V_{\mathcal{N}}$ such that all the non trivial cycles $\mathcal{A}_i$ are in the physical sheet, and such that the cycles $\mathcal{A}_i$ are all cuts. The filling fractions in that case are the $\mathcal{A}$–cycles integrals:

$$\epsilon_i := -\frac{1}{2i\pi}\oint_{\mathcal{A}_i} z_{-1}dz_0\,. \tag{4.23}$$

For more generic potentials, we define the filling fractions by eq. (4.23), eventhough they don't really correspond to numbers of eigenvalues. The cuts are integer homological linear combination of $\mathcal{A}$–cycles.

We also define the chemical potentials as the $\mathcal{B}$–cycle integrals:

$$\Gamma_i := -\oint_{\mathcal{B}_i} z_{-1}dz_0\,. \tag{4.24}$$

We will see in part 6.1 that this description remains valid for the densities of all matrices M_k (and not only M_0). We will prove (see eq. (6.3)) that $\rho_k(z_k)$ the density of eigenvalues of M_k is the discontinuity of z_{k+1} (or equivalently z_{k-1}) along the cuts.

It is easy to prove, by integration by parts, that:

$$\oint_{\mathcal{A}_i} z_{-1}dz_0 = \oint_{\mathcal{A}_i} z_{k-1}dz_k = -\oint_{\mathcal{A}_i} z_{k+1}dz_k \tag{4.25}$$

$$\oint_{\mathcal{B}_i} z_{-1}dz_0 = \oint_{\mathcal{B}_i} z_{k-1}dz_k = -\oint_{\mathcal{B}_i} z_{k+1}dz_k\,. \tag{4.26}$$

If we anticipate on section 6.1, this proves that the filling fractions are the same for all densities, up to integer linear combinations. If the support of the density of eigenvalues of

M_k has $m_k \le g+1$ connected components $[a_{k,i}, b_{k,i}]$ $(i = 1, \dots, m_k)$, the filling fractions in each connected component are:

$$\int_{a_{k,i}}^{b_{k,i}} \rho_k(z_k) dz_k = \frac{1}{T} \sum_{j=1}^{g+1} A_{k,i,j}\, \epsilon_j \tag{4.27}$$

where the coefficients $A_{k,i,j}$ are integers (possibly nul or negative).

4.5 $\mathcal{A}$ or $\mathcal{B}$ cycles fixed ?

As we said before, the loop equation is not sufficient to determine all the unknown coefficients of E. Some additional hypothesis are needed, two of them are often considered in the litterature:

- **condition B**: if all the potentials are bounded from below, and the partition function is well defined, one is interested in finding large-N limits for the free energy and various expectation values of traces of powers of the matrices. In that approach, one has to find a solution of the loop equation which gives an absolute minimum of the free energy. The genus g and the filling fractions ϵ_i are not known, they are determined by the minimization condition $\partial F/\partial \epsilon_i = 0$, which reduces to:

$$\forall i = 1, \dots, g \qquad \oint_{\mathcal{B}_i} z_1 dz_0 = 0 \tag{4.28}$$

  All the $\mathcal{B}$ cycle integrals must vanish. That condition is sufficient to determine g and all the ϵ_i, and it is sufficient to determine the unknown coefficients of E.

  This allows to find the large-N limit of the free energy. Subleading large-N corrections, i.e. the so-called topological large-N expansion of the free energy, exist only if $g = 0$, as was shown in [6, 10]. If the genus g is ≥ 1, there exist asymptotics for the subleading corrections to the free energy, which have no $1/N$ expansion, they are oscillating functions.

- **condition A**: if one is only interested in the formal large-N expansion of the free energy, one has to find a solution of the loop equation which corresponds to the perturbation of a given local extremum of the free energy. Therefore, g and the ϵ_i's are fixed parameters which characterize the minimum around which we perform the perturbative expansion (moduli). The following equations

$$\forall i = 1, \dots, g \qquad \oint_{\mathcal{A}_i} z_1 dz_0 = 2i\pi\epsilon_i \tag{4.29}$$

  are sufficient to determine all the unknown coefficients of E.

In the following, we will always assume that we have condition A, unless specified. Most of the results obtained for condition A, immediately translate to condition B, by exchanging the roles of $\mathcal{A}$ and $\mathcal{B}$-cycles, like the large-N free energy. But some quantities, like the large-N 2–loop functions, get contributions from the oscillating asymptotics, and will not be computed in this article. So, from now on, we assume condition A.

4.6 Parametrization

It is possible to parametrize the algebraic curve $\mathcal{E}$ in terms of θ functions.

Given g and a canonical basis of non-trivial cycles $\mathcal{A}_i, \mathcal{B}_i (i = 1, \dots, g)$, it is known [16, 17] that there exists a unique basis of holomorphic 1-forms du_i on $\mathcal{E}$ such that:

$$\oint_{\mathcal{A}_i} du_j = \delta_{ij} \, . \tag{4.30}$$

The matrix of $\mathcal{B}$ periods is defined as:

$$\tau_{ij} := \oint_{\mathcal{B}_i} du_j = \tau_{ji} \tag{4.31}$$

τ_{ij} is a symmetric $g \times g$ matrix, with positive imaginary part.

If we choose an arbitrary basepoint $p_0 \in \mathcal{E}$, we define the Abel map:

$$\vec{u}(p) := \int_{p_0}^{p} d\vec{u} \tag{4.32}$$

which defines an embedding of $\mathcal{E}$ into $\mathbf{C}^g$.

For any $0 \leq k \leq \mathcal{N}$, $z_k(p)$ is a function on $\mathcal{E}$, with a pole of degree r_k at $p_{\infty+}$, and a pole of degree s_k at $p_{\infty-}$. z_k must have $r_k + s_k$ zeroes, which we denote $p_{k,i}(0)$, $(i = 1, \dots, r_k + s_k)$. The zeroes must satisfy:

$$\sum_{i=1}^{r_k+s_k} \vec{u}(p_{k,i}(0)) = r_k \vec{u}(p_{\infty+}) + s_k \vec{u}(p_{\infty-}) \tag{4.33}$$

then:

$$z_k(p) = A_k \frac{\prod_{i=1}^{r_k+s_k} \theta(\vec{u}(p) - \vec{u}(p_{k,i}(0)) - \vec{z}; \tau)}{\theta(\vec{u}(p) - \vec{u}(p_{\infty+}) - \vec{z}; \tau)^{r_k} \theta(\vec{u}(p) - \vec{u}(p_{\infty-}) - \vec{z}; \tau)^{s_k}} \tag{4.34}$$

where θ is the Riemann theta function [16, 17], $\vec{z}$ is an arbitrary non-singular zero of θ. The ratio of the r.h.s. and the l.h.s. is a function on $\mathcal{E}$ with no pole, therefore it must be a constant, which we call A_k.

We have a parametrization of $\mathcal{E}$ in terms of θ functions.

4.7 Two-loop functions and Bergmann kernel

4.7.1 The Bergmann kernel

The Bergmann kernel $B(p; p')$ is the unique bilinear differential form on $\mathcal{E} \times \mathcal{E}$ with the following properties:

- $B(p; p')$, as a function of p, is a meromorphic form, with only one double pole at $p = p'$ with no residue, and such that in any local parameter $z(p)$ we have:

$$B(p; p') \underset{p \to p'}{\sim} \frac{dz(p) dz(p')}{(z(p) - z(p'))^2} \, ; \tag{4.35}$$

- $\forall i = 1, \dots, g \qquad \oint_{p \in \mathcal{A}_i} B(p, p') = 0.$

4.7.2 The 2-point function

For $0 \leq k \leq \mathcal{N}$ and $0 \leq l \leq \mathcal{N}$, define the following meromorphic differential forms on $\mathcal{E} \times \mathbf{C}$:

$$B_{k,l}(p;z) := \left.\frac{\partial z_{k-1}(p)}{\partial V_l(z)}\right|_{z_k(p)} dz_k(p)\, dz \tag{4.36}$$

$$\tilde{B}_{k,l}(p;z) := \left.\frac{\partial z_{k+1}(p)}{\partial V_l(z)}\right|_{z_k(p)} dz_k(p)\, dz\,. \tag{4.37}$$

We have the "thermodynamic identity":

$$\tilde{B}_{k,l}(p;z) = -B_{k+1,l}(p;z) \tag{4.38}$$

and $z_{k-1} + z_{k+1} = V_k'(z_k)$ implies:

$$B_{k,l}(p;z) + \tilde{B}_{k,l}(p;z) = -\delta_{k,l}\frac{dz_l(p)\, dz}{(z - z_l(p))^2}\,. \tag{4.39}$$

We are going to prove below that:

$$\begin{aligned}
B_{k,l}(p;z) &= -\sum_{j=1}^{s_l} B(p;p_{-j,l}(z)) \quad 0 \leq k \leq l \\
B_{k,l}(p;z) &= \sum_{j=1}^{r_l} B(p;p_{+j,l}(z)) \quad l < k \leq \mathcal{N} \\
\tilde{B}_{k,l}(p;z) &= \sum_{j=1}^{s_l} B(p;p_{-j,l}(z)) \quad 0 \leq k < l \\
\tilde{B}_{k,l}(p;z) &= -\sum_{j=1}^{r_l} B(p;p_{+j,l}(z)) \quad l \leq k \leq \mathcal{N}
\end{aligned} \tag{4.40}$$

where B is the Bergmann kernel, and the $p_{j,l}$'s have been defined in 4.3.

Proof of eq. (4.40). First notice that eq. (4.38) and eq. (4.39) imply:

$$\begin{aligned}
B_{k,l}(p;z) &= B_{0,l}(p;z) \quad 0 \leq k \leq l \\
B_{k,l}(p;z) &= -\tilde{B}_{\mathcal{N},l}(p;z) \quad l < k \leq \mathcal{N}
\end{aligned} \tag{4.41}$$

and:

$$B_{0,l}(p;z) + \tilde{B}_{\mathcal{N},l}(p;z) = -\frac{dz_l(p)\, dz}{(z - z_l(p))^2} \tag{4.42}$$

Consider the two-loop functions introduced in eq. (2.24):

$$W_{0;l}(z;z') = \left\langle \mathrm{tr}\,\frac{1}{z-M_0}\,\mathrm{tr}\,\frac{1}{z'-M_l}\right\rangle_{\mathrm{conn}} = \frac{B_{0,l}(p_{+1,0}(z);z')}{dz_0(p_{+1,0}(z))dz'} \tag{4.43}$$

$$W_{\mathcal{N};l}(z;z') = \left\langle \mathrm{tr}\,\frac{1}{z-M_{\mathcal{N}}}\,\mathrm{tr}\,\frac{1}{z'-M_l}\right\rangle_{\mathrm{conn}} = \frac{\tilde{B}_{\mathcal{N},l}(p_{-1,\mathcal{N}}(z);z')}{dz_{\mathcal{N}}(p_{-1,\mathcal{N}}(z))dz'}\,. \tag{4.44}$$

The expectation values $\left\langle \mathrm{tr}\,\frac{1}{z-M_0}\,\mathrm{tr}\,\frac{1}{z'-M_l}\right\rangle$ and $\left\langle \mathrm{tr}\,\frac{1}{z-M_{\mathcal{N}}}\,\mathrm{tr}\,\frac{1}{z'-M_l}\right\rangle$ are well defined integrals for z in the physical sheet, therefore $B_{0,l}(p;z')$ can have no pole when $z' = z_l(p)$ if p

is in the z_0-physical sheet, and $\tilde{B}_{\mathcal{N},l}(p;z')$ can have no pole when $z' = z_l(p)$ if p is in the $z_{\mathcal{N}}$-physical sheet. Note that the derivatives $\partial/\partial V_l(z')$ are formally defined only for large z', which implies that:

- $B_{0,l}(p;z')$ can have no pole when $p = p_{+j,l}(z')$, $j = 1,\dots,r_l$,
- $\tilde{B}_{\mathcal{N},l}(p;z')$ can have no pole when $p = p_{-j,l}(z')$, $j = 1,\dots,s_l$.

Since the r.h.s. of eq. (4.42) has poles at all $p = p_{\pm j,l}(z')$, we must have:

- $B_{0,l}(p;z')$ can have no pole when $p = p_{+j,l}(z')$, $j = 1,\dots,r_l$,
- $B_{0,l}(p;z')$ has double poles when $p = p_{-j,l}(z')$, $j = 1,\dots,s_l$.

That implies that:

$$\overline{B}_l(p;z') := B_{0,l}(p;z') + \sum_{j=1}^{s_l} B(p;p_{-j,l}(z')) \tag{4.45}$$

has no pole when $z_l(p) = z'$.

It obeys the following properties:

- Since $W_0(z_0)$ behaves as $1/z_0$ near ∞ in the physical sheet, $W_{0;l}(z_0;z')$ must behave as $O(1/z_0^2)$ near $p_{\infty+}$, while dz_0 has a double pole at $p_{\infty+}$. Therefore $B_{0,l}(p;z')$ has no pole when $p \to p_{\infty+}$, and thus $\overline{B}_l(p;z')$ has no pole when $p \to p_{\infty+}$.
- Similarly, $\tilde{B}_{\mathcal{N},l}(p;z')$ has no pole when $p \to p_{\infty-}$, and with eq. (4.42), that implies that $B_{0,l}(p;z')$ has no pole when $p \to p_{\infty-}$, and thus $\overline{B}_l(p;z')$ has no pole when $p \to p_{\infty-}$.
- Near an endpoint $e_{k,i}$, $Z_{k-1}(z_k)$ has a square root singularity, i.e. the derivative $\partial z_{k-1}(p)/\partial V_l(z')|_{z_k(p)}$ has an inverse square root singularity, i.e. a simple pole which is exactly compensated by the zero of $dz_k(p)$. Therefore $B_{k,l}(p;z')$ has no pole when $p \to e_{k,i}$, and so for $\overline{B}_l(p;z')$.
- Near any other point, z_{k-1} is an analytical function of z_k, and thus $B_{k,l}$ is analytical.
- $B_{k,l}$ must satisfy:

$$\oint_{p\in\mathcal{A}_i} B_{k,l}(p;z') = 0 \tag{4.46}$$

  and so must $\overline{B}_l$.

Finaly we find that $\overline{B}_l$ is a meromorphic form with no pole, with all its $\mathcal{A}$ cycle integrals vanishing, therefore: $\overline{B}_l = 0$. □

Remark. We have:

$$B(p,p') = d_p d_{p'} \ln\theta(\vec{u}(p) - \vec{u}(p') - \vec{z})\,. \tag{4.47}$$

Notice that

$$du_i(p') = \frac{1}{2i\pi}\oint_{p\in\mathcal{B}_i} B(p,p') = dz_0(p')\frac{1}{2i\pi}\frac{\partial\Gamma_i}{\partial V_0(z_0(p'))}\,. \tag{4.48}$$

Once B is known, eq. (4.48) give an explicit way of computing the Abel map $\vec{u}(p)$.

4.8 Abelian differential of the 3rd kind and temperature

4.8.1 Abelian differential of the 3rd kind

There is a unique abelian differential dS on $\mathcal{E}$ with only two simple poles at $p_{\infty\pm}$, such that:

$$\mathop{\mathrm{Res}}_{p_{\infty+}} dS = -\mathop{\mathrm{Res}}_{p_{\infty-}} dS = 1\,, \qquad \forall i=1,\dots,g \quad \oint_{\mathcal{A}_i} dS = 0 \tag{4.49}$$

It has the property that:

$$dS(p) = \int_{p'=p_{\infty-}}^{p_{\infty+}} B(p,p')\,. \tag{4.50}$$

We define:

$$\eta_i := \oint_{\mathcal{B}_i} dS\,. \tag{4.51}$$

And, given a basepoint p_0, we define the following multivalued function on $\mathcal{E}$:

$$\Lambda(p) := \mathrm{e}^{-\int_{p_0}^{p} dS} \tag{4.52}$$

Λ has a simple pole at $p_{\infty+}$ and a single zero at $p_{\infty-}$, therefore the following quantities are well defined:

$$\gamma := \lim_{p\to p_{\infty+}} \frac{z_0(p)}{\Lambda(p)}\,, \quad \tilde\gamma := \lim_{p\to p_{\infty-}} z_{\mathcal{N}}(p)\Lambda(p)\,. \tag{4.53}$$

Notice that the product $\gamma\tilde\gamma$ is independent of the choice of the basepointp_0.

4.8.2 Derivatives with respect to T

Consider the abelian differentials:

$$dS_k(p) := \left.\frac{\partial z_{k-1}(p)}{\partial T}\right|_{z_k(p)} dz_k(p)\,, \quad d\tilde S_k(p) := \left.\frac{\partial z_{k+1}(p)}{\partial T}\right|_{z_k(p)} dz_k(p)\,. \tag{4.54}$$

We clearly have (from $z_{k+1}+z_{k-1}=V_k'(z_k)$ and thermodynamic identity):

$$dS_k(p) = -d\tilde S_k(p) = dS_{k+1}(p)\,. \tag{4.55}$$

Therefore dS_k and $d\tilde S_k$ are independent of k.

They have the following properties:

- near $p_{\infty+}$, $z_{-1}\sim T/z_0 + O(z_0^{-2})$, thus $dS_0 \sim dz_0/z_0$, i.e. dS_0 has a single pole at $p_{\infty+}$, with residue -1.
- Similarly $d\tilde S_{\mathcal{N}}$ has a single pole at $p_{\infty-}$, with residue -1, i.e. dS_0 has a single pole at $p_{\infty-}$, with residue $+1$.
- near an endpoint $e_{k,i}$, $Z_{k-1}(z_k)$ has a square root singularity, thus $\left.\frac{\partial z_{k-1}}{\partial T}\right|_{z_k}$ has an inverse square root singularity, i.e. a single pole at $e_{k,i}$, which is compensated by the single zero of dz_K, and therefore dS_0 has no pole at $e_{k,i}$.
- near any other point, z_k is a local parameter and z_{k+1} is analytical in z_k, i.e. dS_0 has no pole.

- The $\mathcal{A}_i$ cycle integrals of dS_0 vanish.

There is a unique abelian differential with such properties, it is the abelian differential of the third kind defined in eq. (4.49):

$$d\tilde{S}_k(p) = -dS_k(p) = dS(p)\,. \tag{4.56}$$

In particular, we have that:

$$\frac{\partial}{\partial T}\Gamma_i = \eta_i\,, \quad \frac{\partial}{\partial T}\epsilon_i = 0\,. \tag{4.57}$$

4.9 Derivatives with respect to ϵ_i (condition A)

Consider:

$$du_{k,i}(p) := \frac{1}{2i\pi}\left.\frac{\partial z_{k-1}(p)}{\partial \epsilon_i}\right|_{z_k(p)} dz_k(p)\,, \quad d\tilde{u}_{k,i}(p) := \frac{1}{2i\pi}\left.\frac{\partial z_{k+1}(p)}{\partial \epsilon_i}\right|_{z_k(p)} dz_k(p)\,. \tag{4.58}$$

We clearly have (from $z_{k+1} + z_{k-1} = V_k'(z_k)$ and thermodynamic identity):

$$du_{k,i}(p) = -d\tilde{u}_{k,i}(p) = du_{k+1,i}(p)\,. \tag{4.59}$$

Therefore $du_{k,i}$ and $d\tilde{u}_{k,i}$ are independent of k.

Following the same lines as in the previous section, we show that $du_{k,i}$ and $d\tilde{u}_{k,i}$ have no poles, i.e. they are holomorphic one-forms. Moreover we have:

$$\oint_{\mathcal{A}_j} d\tilde{u}_{k,i} = \frac{\partial \epsilon_j}{\partial \epsilon_i} = \delta_{i,j}\,. \tag{4.60}$$

There exists a unique set of holomorphic one-forms with those properties, it is the holomorphic forms du_i introduced in eq. (4.30), therefore:

$$d\tilde{u}_{k,i} = -du_{k,i} = du_i\,. \tag{4.61}$$

This shows that:

$$\frac{\partial \Gamma_j}{\partial \epsilon_i} = 2i\pi\,\tau_{i,j} \tag{4.62}$$

where $\tau_{i,j}$ is the matrix of periods introduced in eq. (4.31).

5. Large N free energy

5.1 The large-N free energy

The free energy F defined in eq. (1.2) has a large-N limit:

$$F = F^{(0)} + O(N^{-2})\,. \tag{5.1}$$

We prove below, a generalization of the formula of [3]:

$$\boxed{\begin{array}{rl} 2F^{(0)} & = \sum_{k=0}^{\mathcal{N}} \mathop{\rm Res}_{p_{\infty_+}} (V_k(z_k) - \frac{1}{2} z_k V'_k(z_k)) z_{k+1} dz_k \\ & \quad + T\mu + \sum_i \epsilon_i \Gamma_i - T^2(1+\mathcal{N}\ln T) \\ & = \sum_{k=0}^{\mathcal{N}} \mathop{\rm Res}_{p_{\infty_-}} (V_k(z_k) - \frac{1}{2} z_k V'_k(z_k)) z_{k-1} dz_k \\ & \quad + T\mu + \sum_i \epsilon_i \Gamma_i - T^2(1+\mathcal{N}\ln T) \end{array}} \tag{5.2}$$

where μ is the generalized version of [3] and is defined as follows for any $p \in \mathcal{E}$:

$$\begin{aligned} \mu := & \int_{p_{\infty+}}^{p} \left(\frac{T}{z_0} - z_{-1} \right) dz_0 + \int_{p_{\infty-}}^{p} \left(\frac{T}{z_{\mathcal{N}}} - z_{\mathcal{N}+1} \right) dz_{\mathcal{N}} - T \ln z_0(p) - T \ln z_{\mathcal{N}}(p) + \\ & + \sum_{k=0}^{\mathcal{N}} V_k(z_k(p)) - \sum_{k=1}^{\mathcal{N}} z_{k-1}(p) z_k(p) \end{aligned} \tag{5.3}$$

μ is independent of p (indeed $d\mu$ is a telescopic sum which cancels completely).

Proof of eq. (5.2): we remind that we assume condition A. We define:

$$\begin{aligned} 4K := & 2T\mu + 2\sum_i \epsilon_i \Gamma_i - 4F^{(0)} + \sum_{k=0}^{\mathcal{N}} \mathop{\rm Res}_{p_{\infty+}} \left(V_k(z_k) - \frac{1}{2} z_k z_{k+1} \right) z_{k+1} dz_k + \\ & + \sum_{k=0}^{\mathcal{N}} \mathop{\rm Res}_{p_{\infty-}} \left(V_k(z_k) - \frac{1}{2} z_k z_{k-1} \right) z_{k-1} dz_k \,. \end{aligned} \tag{5.4}$$

Notice that $\mathop{\rm Res}_{p_{\infty+}} z_k z_{k-1} z_{k+1} dz_k + \mathop{\rm Res}_{p_{\infty-}} z_k z_{k-1} z_{k+1} dz_k = 0$, so that the expression in eq. (5.4) is the same as in eq. (5.2).

Let us compute:

$$\begin{aligned} 4\frac{\partial K}{\partial V_l(z)} = & \sum_k \mathop{\rm Res}_{p_{\infty+}} \frac{\partial V_k(z_k)}{\partial V_l(z)} z_{k+1} dz_k + \sum_k \mathop{\rm Res}_{p_{\infty-}} \frac{\partial V_k(z_k)}{\partial V_l(z)} z_{k-1} dz_k + 4TW_l(z) + \\ & + \sum_k \mathop{\rm Res}_{p_{\infty+}} \left.\frac{\partial z_{k+1}}{\partial V_l(z)}\right|_{z_k} (V_k(z_k) - z_k z_{k+1}) dz_k + \\ & + \sum_k \mathop{\rm Res}_{p_{\infty-}} \left.\frac{\partial z_{k-1}}{\partial V_l(z)}\right|_{z_k} (V_k(z_k) - z_k z_{k-1}) dz_k + \\ & + 2T \frac{\partial \mu}{\partial V_l(z)} + 2\sum_i \epsilon_i \frac{\partial \Gamma_i}{\partial V_l(z)} \end{aligned} \tag{5.5}$$

we introduce the following multivalued functions:

$$\zeta_k(p) := \int_{p_{\infty+}}^{p} \left.\frac{\partial z_{k+1}}{\partial V_l(z)}\right|_{z_k} dz_k \,, \quad \tilde{\zeta}_k(p) := \int_{p_{\infty-}}^{p} \left.\frac{\partial z_{k-1}}{\partial V_l(z)}\right|_{z_k} dz_k \,. \tag{5.6}$$

By integration by parts, and using eq. (2.17), we have:

$$
\begin{aligned}
4\frac{\partial K}{\partial V_l(z)} &= -\mathop{\mathrm{Res}}_{p_{\infty+}} \frac{z_{l+1}}{z-z_l} dz_l - \mathop{\mathrm{Res}}_{p_{\infty-}} \frac{z_{l-1}}{z-z_l} dz_l + 4T W_l(z) - \\
&\quad - \sum_k \mathop{\mathrm{Res}}_{p_{\infty+}} \zeta_k ((V_k'(z_k) - z_{k+1}) dz_k - z_k dz_{k+1}) - \\
&\quad - \sum_k \mathop{\mathrm{Res}}_{p_{\infty-}} \tilde{\zeta}_k ((V_k'(z_k) - z_{k-1}) dz_k - z_k dz_{k-1}) + 2T \frac{\partial \mu}{\partial V_l(z)} + 2\sum_i \epsilon_i \frac{\partial \Gamma_i}{\partial V_l(z)} \\
&= -\mathop{\mathrm{Res}}_{p_{\infty+}} \frac{z_{l+1}}{z-z_l} dz_l - \mathop{\mathrm{Res}}_{p_{\infty-}} \frac{z_{l-1}}{z-z_l} dz_l + 4T W_l(z) - \\
&\quad - \sum_k \mathop{\mathrm{Res}}_{p_{\infty+}} \zeta_k (z_{k-1} dz_k - z_k dz_{k+1}) - \\
&\quad - \sum_k \mathop{\mathrm{Res}}_{p_{\infty-}} \tilde{\zeta}_k (z_{k+1} dz_k - z_k dz_{k-1}) + 2T \frac{\partial \mu}{\partial V_l(z)} + 2\sum_i \epsilon_i \frac{\partial \Gamma_i}{\partial V_l(z)} \\
&= -\mathop{\mathrm{Res}}_{p_{\infty+}} \frac{z_{l+1}}{z-z_l} dz_l - \mathop{\mathrm{Res}}_{p_{\infty-}} \frac{z_{l-1}}{z-z_l} dz_l + 4T W_l(z) + \\
&\quad + \mathop{\mathrm{Res}}_{p_{\infty+}} \zeta_{\mathcal{N}} z_{\mathcal{N}} dz_{\mathcal{N}+1} - \zeta_{-1} z_{-1} dz_0 - \sum_{k=0}^{\mathcal{N}} (\zeta_k - \zeta_{k-1}) z_{k-1} dz_k \\
&\quad + \mathop{\mathrm{Res}}_{p_{\infty-}} \tilde{\zeta}_0 z_0 dz_{-1} - \tilde{\zeta}_{\mathcal{N}+1} z_{\mathcal{N}+1} dz_{\mathcal{N}} - \sum_{k=0}^{\mathcal{N}} (\tilde{\zeta}_k - \tilde{\zeta}_{k+1}) z_{k+1} dz_k \\
&\quad + 2T \frac{\partial \mu}{\partial V_l(z)} + 2\sum_i \epsilon_i \frac{\partial \Gamma_i}{\partial V_l(z)} \,. \qquad (5.7)
\end{aligned}
$$

Near $p_{\infty+}$ we have $z_{-1} \sim T/z_0 + O(z_0^{-2})$, therefore $z_0 \sim T/z_{-1} + O(1)$, and thus ζ_{-1} has a zero at $p_{\infty+}$. That implies that:

$$
\mathop{\mathrm{Res}}_{p_{\infty+}} \zeta_{-1} z_{-1} dz_0 = 0\,, \quad \mathop{\mathrm{Res}}_{p_{\infty-}} \tilde{\zeta}_{\mathcal{N}+1} z_{\mathcal{N}+1} dz_{\mathcal{N}} = 0\,. \qquad (5.8)
$$

Moreover, notice that $\left.\frac{\partial z_k}{\partial V_l(z)}\right|_{z_{k-1}} dz_{k-1} = -\left.\frac{\partial z_{k-1}}{\partial V_l(z)}\right|_{z_k} dz_k$, therefore:

$$
\zeta_k(p) - \zeta_{k-1}(p) = \int_{p_{\infty+}}^{p} \left(\left.\frac{\partial z_{k+1}}{\partial V_l(z)}\right|_{z_k} + \left.\frac{\partial z_{k-1}}{\partial V_l(z)}\right|_{z_k} \right) dz_k = \int_{p_{\infty+}}^{p} \frac{\partial V_k'(z_k)}{\partial V_l(z)} dz_k = \frac{\delta_{k,l}}{z_l(p) - z} \,. \qquad (5.9)
$$

Thus:

$$
\begin{aligned}
4\frac{\partial K}{\partial V_l(z)} &= \mathop{\mathrm{Res}}_{p_{\infty+}} \frac{z_{l-1} - z_{l+1}}{z-z_l} dz_l + \mathop{\mathrm{Res}}_{p_{\infty-}} \frac{z_{l+1} - z_{l-1}}{z-z_l} dz_l + 4T W_l(z) + \\
&\quad + \mathop{\mathrm{Res}}_{p_{\infty+}} \zeta_{\mathcal{N}} z_{\mathcal{N}} dz_{\mathcal{N}+1} + \mathop{\mathrm{Res}}_{p_{\infty-}} \tilde{\zeta}_0 z_0 dz_{-1} + 2T \frac{\partial \mu}{\partial V_l(z)} + 2\sum_i \epsilon_i \frac{\partial \Gamma_i}{\partial V_l(z)} \\
&= \mathop{\mathrm{Res}}_{p_{\infty+}} \frac{z_{l-1} - z_{l+1}}{z-z_l} dz_l + \mathop{\mathrm{Res}}_{p_{\infty-}} \frac{z_{l+1} - z_{l-1}}{z-z_l} dz_l + 4T W_l(z) - \\
&\quad - \mathop{\mathrm{Res}}_{p_{\infty-}} \zeta_{\mathcal{N}} z_{\mathcal{N}} dz_{\mathcal{N}+1} - \mathop{\mathrm{Res}}_{p_{\infty+}} \tilde{\zeta}_0 z_0 dz_{-1} + 2T \frac{\partial \mu}{\partial V_l(z)} + 2\sum_i \epsilon_i \frac{\partial \Gamma_i}{\partial V_l(z)} + \\
&\quad + \sum_i \frac{1}{2i\pi} \oint_{\mathcal{A}_i} \mathrm{disc}_{\mathcal{A}_i} (\tilde{\zeta}_0 z_0 dz_{-1}) + \sum_i \frac{1}{2i\pi} \oint_{\mathcal{A}_i} \mathrm{disc}_{\mathcal{A}_i} (\zeta_{\mathcal{N}} z_{\mathcal{N}} dz_{\mathcal{N}+1}) -
\end{aligned}
$$

$$-\sum_i \frac{1}{2i\pi}\oint_{\mathcal{B}_i} \mathrm{disc}_{\mathcal{B}_i}\,(\tilde{\zeta}_0 z_0 dz_{-1}) - \sum_i \frac{1}{2i\pi}\oint_{\mathcal{B}_i} \mathrm{disc}_{\mathcal{B}_i}\,(\zeta_{\mathcal{N}} z_{\mathcal{N}} dz_{\mathcal{N}+1}) \tag{5.10}$$

where we have used Riemann bilinear identity, and disc means taking discontinuity of the considered multivalued function when crossing cycles.

Now compute $\partial\mu/\partial V_l(z)$, all terms cancel but:

$$\frac{\partial\mu}{\partial V_l(z)} = -\frac{1}{z-z_l(p)} - \int_{p_{\infty+}}^{p} \left.\frac{\partial z_{-1}}{\partial V_l(z)}\right|_{z_0} dz_0 - \int_{p_{\infty-}}^{p} \left.\frac{\partial z_{\mathcal{N}+1}}{\partial V_l(z)}\right|_{z_{\mathcal{N}}} dz_{\mathcal{N}} \tag{5.11}$$

which is independent of p. In particular for $p = p_{\infty\pm}$, this proves that $\tilde{\zeta}_0(p_{\infty+})$ and $\zeta_{\mathcal{N}}(p_{\infty-})$ are finite, and:

$$\frac{\partial\mu}{\partial V_l(z)} = \tilde{\zeta}_0(p_{\infty+}) = \zeta_{\mathcal{N}}(p_{\infty-})\,. \tag{5.12}$$

Thus:

$$\mathop{\mathrm{Res}}_{p_{\infty+}} \tilde{\zeta}_0 z_0 dz_{-1} = T\tilde{\zeta}_0(p_{\infty+}) = T\frac{\partial\mu}{\partial V_l(z)} = T\zeta_{\mathcal{N}}(p_{\infty-}) = \mathop{\mathrm{Res}}_{p_{\infty-}} \zeta_{\mathcal{N}} z_{\mathcal{N}} dz_{\mathcal{N}+1}\,. \tag{5.13}$$

Now compute the discontinuities (crossing $\mathcal{B}_i$ is equivalent to going around $\mathcal{A}_i$):

$$\mathrm{disc}_{\mathcal{B}_i}\,\tilde{\zeta}_0 = \oint_{\mathcal{A}_i} \left.\frac{\partial z_{-1}}{\partial V_l(z)}\right|_{z_0} dz_0 = -2i\pi\frac{\partial\epsilon_i}{\partial V_l(z)} = 0 \tag{5.14}$$

this implies that $\tilde{\zeta}_0$ has no discontinuity along $\mathcal{B}_i$, and

$$\mathrm{disc}_{\mathcal{A}_i}\,\tilde{\zeta}_0 = \oint_{\mathcal{B}_i} \left.\frac{\partial z_{-1}}{\partial V_l(z)}\right|_{z_0} dz_0 = -\frac{\partial\Gamma_i}{\partial V_l(z)} \tag{5.15}$$

this implies that $\tilde{\zeta}_0$ has a constant discontinuity along $\mathcal{A}_i$. Moreover z_0 and dz_{-1} are monovalued, i.e. they have no discontinuity along $\mathcal{A}_i$ or $\mathcal{B}_i$. Thus:

$$\oint_{\mathcal{A}_i} \mathrm{disc}_{\mathcal{A}_i}\,(\tilde{\zeta}_0 z_0 dz_{-1}) = -\frac{\partial\Gamma_i}{\partial V_l(z)}\oint_{\mathcal{A}_i} z_0 dz_{-1} = -2i\pi\epsilon_i\frac{\partial\Gamma_i}{\partial V_l(z)}\,. \tag{5.16}$$

Using similar arugments for $z_{\mathcal{N}}$, we arrive at:

$$4\frac{\partial K}{\partial V_l(z)} = \mathop{\mathrm{Res}}_{p_{\infty+}} \frac{z_{l-1}-z_{l+1}}{z-z_l}dz_l + \mathop{\mathrm{Res}}_{p_{\infty-}} \frac{z_{l+1}-z_{l-1}}{z-z_l}dz_l + 4TW_l(z)\,. \tag{5.17}$$

When $l=0$, by definition eq. (2.8), we have

$$\begin{aligned} TW_0(z) &= \mathop{\mathrm{Res}}_{p_{\infty+}} \frac{1}{z-z_0} z_1 dz_0 = -\mathop{\mathrm{Res}}_{p_{\infty+}} \frac{1}{z-z_0} z_{-1} dz_0 \\ &= \mathop{\mathrm{Res}}_{p_{\infty-}} \frac{1}{z-z_0} z_{-1} dz_0 = -\mathop{\mathrm{Res}}_{p_{\infty-}} \frac{1}{z-z_0} z_1 dz_0 \end{aligned} \tag{5.18}$$

therefore:

$$\frac{\partial K}{\partial V_0(z)} = 0 \tag{5.19}$$

which proves that K is independent of V_0. In particular we can choose V_0 quadratic, and then we integrate M_0 out, i.e. we reduce the problem of a chain of lenght $\mathcal{N}$ with potentials $V_0,\dots,V_{\mathcal{N}}$ to a chain of lenght $\mathcal{N}-1$ with potentials $V_1-\frac{z_1^2}{2},\dots,V_{\mathcal{N}}$. It is easy to check directly that:

$$F_{\mathcal{N}}(\frac{z_0^2}{2},V_1,\dots,V_{\mathcal{N}}) = F_{\mathcal{N}-1}(V_1-\frac{z_1^2}{2},\dots,V_{\mathcal{N}}) - \frac{T^2}{2}\ln T \tag{5.20}$$

and thus:

$$K_{\mathcal{N}}(V_0,\dots,V_{\mathcal{N}}) = K_{\mathcal{N}}(\frac{z_0^2}{2},V_1,\dots,V_{\mathcal{N}}) = K_{\mathcal{N}-1}(V_1-\frac{z_1^2}{2},\dots,V_{\mathcal{N}}) + \frac{T^2}{2}\ln T\,. \tag{5.21}$$

Therefore, by recursion on $\mathcal{N}$, we find that K is independent of all $V's$. K could still depend on T and the ϵ_i's. K has been computed for $\mathcal{N}=1$ and $\mathcal{N}=0$ [3], and we find:

$$K = \frac{T^2}{2}(1+\mathcal{N}\ln T) \tag{5.22}$$

which is the same result as if all potentials are chosen gaussian (see section 8.2). □

Remark. En route, we have proved that for all l, the resolvent of the l^{th} matrix is:

$$\begin{aligned} W_l(z) &= \frac{1}{T}\mathop{\mathrm{Res}}_{p_{\infty+}} \frac{1}{z-z_l} z_{l+1}dz_l = -\frac{1}{T}\mathop{\mathrm{Res}}_{p_{\infty+}} \frac{1}{z-z_l} z_{l-1}dz_l \\ &= \frac{1}{T}\mathop{\mathrm{Res}}_{p_{\infty-}} \frac{1}{z-z_l} z_{l-1}dz_l = -\frac{1}{T}\mathop{\mathrm{Res}}_{p_{\infty-}} \frac{1}{z-z_l} z_{l+1}dz_l \end{aligned} \tag{5.23}$$

Remark. we have:

$$\mu = -T\ln\gamma\tilde{\gamma} + \sum_i \epsilon_i\eta_i + \mathop{\mathrm{Res}}_{p_{\infty+}}\left(\sum_{k=0}^{\mathcal{N}} V_k(z_k) - \sum_{k=1}^{\mathcal{N}} z_{k-1}z_k\right)dS \tag{5.24}$$

where dS and γ and $\tilde{\gamma}$ and η_i are defined in eq. (4.49) and eq. (4.53).
Proof of eq. (5.24): Consider the functions:

$$\phi_0(p) := \int_{p_{\infty+}}^{p} (\frac{T}{z_0} - z_{-1})dz_0\,, \quad \phi_{\mathcal{N}}(p) := \int_{p_{\infty-}}^{p} (\frac{T}{z_{\mathcal{N}}} - z_{\mathcal{N}+1})dz_{\mathcal{N}} \tag{5.25}$$

They satisfy:

$$\mathop{\mathrm{Res}}_{p_{\infty+}} \phi_0 dS = 0\,, \quad \mathop{\mathrm{Res}}_{p_{\infty-}} \phi_{\mathcal{N}} dS = 0 \tag{5.26}$$

and they have the following discontinuities along $\mathcal{A}_i$ or $\mathcal{B}_i$:

$$\mathrm{disc}_{\mathcal{A}_i}\,\phi_0 = -\mathrm{disc}_{\mathcal{A}_i}\,\phi_{\mathcal{N}} = \Gamma_i\,, \quad \mathrm{disc}_{\mathcal{B}_i}\,\phi_0 = -\mathrm{disc}_{\mathcal{B}_i}\,\phi_{\mathcal{N}} = 2i\pi\,\epsilon_i \tag{5.27}$$

From eq. (5.3) and eq. (4.49), we have:

$$\begin{aligned} \mu &= \mathop{\mathrm{Res}}_{p_{\infty+}} \mu dS = \mathop{\mathrm{Res}}_{p_{\infty+}} \phi_0 dS + \mathop{\mathrm{Res}}_{p_{\infty+}} \phi_{\mathcal{N}} dS - T\ln\gamma\tilde{\gamma} \\ &\quad + \mathop{\mathrm{Res}}_{p_{\infty+}}\left(\sum_{k=0}^{\mathcal{N}} V_k(z_k(p)) - \sum_{k=1}^{\mathcal{N}} z_{k-1}(p)z_k(p)\right)dS\,. \end{aligned} \tag{5.28}$$

We need to compute $\mathrm{Res}_{p_{\infty+}} \phi_{\mathcal{N}} dS$, we use Riemann's bilinear identity:

$$\begin{aligned}\mathop{\mathrm{Res}}_{p_{\infty+}} \phi_{\mathcal{N}} dS &= -\mathop{\mathrm{Res}}_{p_{\infty-}} \phi_{\mathcal{N}} dS + \sum_i \frac{1}{2i\pi} \oint_{\mathcal{A}_i} \mathrm{disc}_{\mathcal{A}_i} \phi_{\mathcal{N}} dS - \sum_i \frac{1}{2i\pi} \oint_{\mathcal{B}_i} \mathrm{disc}_{\mathcal{B}_i} \phi_{\mathcal{N}} dS \\ &= \sum_i \eta_i \epsilon_i \,. \end{aligned} \tag{5.29}$$

□

5.2 Derivatives with respect to ϵ_i

We have:

$$\frac{d\mu}{d\epsilon_i} = \eta_i \tag{5.30}$$

indeed, from eq. (5.3), we have:

$$\frac{\partial \mu}{\partial \epsilon_i} = -\int_{p_{\infty+}}^{p} \left.\frac{\partial z_{-1}}{\partial \epsilon_i}\right|_{z_0} dz_0 - \int_{p_{\infty-}}^{p} \left.\frac{\partial z_{\mathcal{N}+1}}{\partial \epsilon_i}\right|_{z_{\mathcal{N}}} dz_{\mathcal{N}} \tag{5.31}$$

using eq. (4.61) we have:

$$\frac{\partial \mu}{\partial \epsilon_i} = 2i\pi \int_{p_{\infty+}}^{p_{\infty-}} du_i = \eta_i \,. \tag{5.32}$$

We have:

$$\frac{dF^{(0)}}{d\epsilon_i} = \Gamma_i \,. \tag{5.33}$$

Indeed, from eq. (5.2), and using eq. (4.61), we have:

$$\begin{aligned} 4\frac{\partial F^{(0)}}{\partial \epsilon_i} &= 2T\eta_i + 2\Gamma_i + 2\sum_j \epsilon_j \tau_{i,j} + \sum_k \mathop{\mathrm{Res}}_{p_{\infty+}} (V_k(z_k) - z_k z_{k+1}) du_i - \\ &\quad - \sum_k \mathop{\mathrm{Res}}_{p_{\infty-}} (V_k(z_k) - z_k z_{k-1}) du_i \,. \end{aligned} \tag{5.34}$$

Let us introduce the multivalued function:

$$u_i(p) := \int_{p_{\infty+}}^{p} du_i \tag{5.35}$$

its discontinuities along the $\mathcal{A}$ and $\mathcal{B}$ cycles are:

$$\mathrm{disc}_{\mathcal{A}_j} u_i = \tau_{i,j} \,. \qquad \mathrm{disc}_{\mathcal{A}_j} u_i = \delta_{i,j} \,. \tag{5.36}$$

After an integration by parts, we have:

$$\begin{aligned} 4\frac{\partial F}{\partial \epsilon_i} &= 2T\eta_i + 2\Gamma_i + 2\sum_j \epsilon_j \tau_{i,j} - \sum_k \mathop{\mathrm{Res}}_{p_{\infty+}} u_i (V_k'(z_k) dz_k - z_{k+1} dz_k - z_k dz_{k+1}) + \\ &\quad + \sum_k \mathop{\mathrm{Res}}_{p_{\infty-}} u_i (V_k'(z_k) dz_k - z_{k-1} dz_k - z_k dz_{k-1}) \\ &= 2T\eta_i + 2\Gamma_i + 2\sum_j \epsilon_j \tau_{i,j} - \sum_k \mathop{\mathrm{Res}}_{p_{\infty+}} u_i (z_{k-1} dz_k - z_k dz_{k+1}) + \end{aligned}$$

$$
\begin{aligned}
&+\sum_k \operatorname*{Res}_{p_{\infty-}} u_i(z_{k+1}dz_k - z_k dz_{k-1}) \\
&= 2T\eta_i + 2\Gamma_i + 2\sum_j \epsilon_j \tau_{i,j} - \operatorname*{Res}_{p_{\infty+}} u_i z_{-1} dz_0 + \operatorname*{Res}_{p_{\infty+}} u_i z_{\mathcal{N}} dz_{\mathcal{N}+1} + \\
&+ \operatorname*{Res}_{p_{\infty-}} u_i z_{\mathcal{N}+1} dz_{\mathcal{N}} - \operatorname*{Res}_{p_{\infty-}} u_i z_0 dz_{-1} \\
&= 2T\eta_i + 2\Gamma_i + 2\sum_j \epsilon_j \tau_{i,j} + \operatorname*{Res}_{p_{\infty+}} u_i z_{\mathcal{N}} dz_{\mathcal{N}+1} + T u_i(p_{\infty-}) - \operatorname*{Res}_{p_{\infty-}} u_i z_0 dz_{-1} \qquad (5.37)
\end{aligned}
$$

using Riemann's bilinear identity and eq. (5.24), we find eq. (5.33).

5.3 Derivatives with respect to T

We have:

$$
\frac{\partial \mu}{\partial T} = -\ln \gamma\tilde{\gamma} \tag{5.38}
$$

indeed, from eq. (5.3), we have:

$$
\frac{\partial \mu}{\partial T} = \int_{p_{\infty+}}^{p} \left(\frac{1}{z_0} - \left.\frac{\partial z_{-1}}{\partial T}\right|_{z_0} \right) dz_0 + \int_{p_{\infty-}}^{p} \left(\frac{1}{z_{\mathcal{N}}} - \left.\frac{\partial z_{\mathcal{N}+1}}{\partial T}\right|_{z_{\mathcal{N}}} \right) dz_{\mathcal{N}} - \ln z_0 - \ln z_{\mathcal{N}} \tag{5.39}
$$

using eq. (4.56) and eq. (4.52), we have:

$$
\begin{aligned}
\frac{\partial \mu}{\partial T} &= \int_{p_{\infty+}}^{p} \left(\frac{dz_0}{z_0} - \frac{d\Lambda}{\Lambda} \right) + \int_{p_{\infty-}}^{p} \left(\frac{dz_{\mathcal{N}}}{z_{\mathcal{N}}} + \frac{d\Lambda}{\Lambda} \right) - \ln z_0 - \ln z_{\mathcal{N}} \\
&= \ln \frac{z_0}{\Lambda} - \ln\gamma + \ln z_{\mathcal{N}}\Lambda - \ln\tilde{\gamma} - \ln z_0 - \ln z_{\mathcal{N}} = -\ln\gamma\tilde{\gamma}\,. \qquad (5.40)
\end{aligned}
$$

We have:

$$
\frac{dF^{(0)}}{dT} = \mu - T\left(\frac{\mathcal{N}+3}{2} + \mathcal{N}\ln T \right). \tag{5.41}
$$

Indeed, from eq. (5.2), and using eq. (4.56), we have:

$$
\begin{aligned}
4\frac{\partial F}{\partial T} &= 2\mu + 2T\frac{\partial \mu}{\partial T} + 2\sum_i \epsilon_i \eta_i + \sum_k \operatorname*{Res}_{p_{\infty+}} (V_k(z_k) - z_k z_{k+1}) dS - \\
&- \sum_k \operatorname*{Res}_{p_{\infty-}} (V_k(z_k) - z_k z_{k-1}) dS - 4T(1+\mathcal{N}\ln T) - 2\mathcal{N}T \qquad (5.42)
\end{aligned}
$$

using eq. (5.38) and eq. (5.24), we find eq. (5.41).

Remark. The derivative of the free energy wrt to T can be computed directely from eq. (1.1):

$$
-N^2 \left.\frac{\partial(F/T^2)}{\partial T}\right|_{\epsilon_i/T} = \frac{N}{T^2} \left\langle \operatorname{tr} \sum_{k=0}^{\mathcal{N}} V_k(M_k) - \sum_{k=1}^{\mathcal{N}} M_k M_{k-1} \right\rangle. \tag{5.43}
$$

The matrix integral is defined for fixed filling fractions, i.e. for fixed ϵ_i/T. We can thus write:

$$
T^2 \frac{\partial(F/T^2)}{\partial T} + \sum_i \frac{\epsilon_i}{T}\frac{\partial F}{\partial \epsilon_i} = -\frac{1}{N} \left\langle \operatorname{tr} \sum_{k=0}^{\mathcal{N}} V_k(M_k) - \sum_{k=1}^{\mathcal{N}} M_k M_{k-1} \right\rangle. \tag{5.44}
$$

Using the change of variable $\delta M_k = M_k$, we get the loop equation:

$$2T = \frac{1}{N}\left\langle \mathrm{tr}\, M_k V_k'(M_k) - M_k(M_{k-1}+M_{k+1})\right\rangle \tag{5.45}$$

which implies:

$$-T^2\frac{\partial(F/T^2)}{\partial T} - \frac{1}{T}\sum_i \epsilon_i\Gamma_i = \frac{1}{N}\left\langle \mathrm{tr}\sum_{k=0}^{\mathcal{N}}(V_k(M_k) - \frac{1}{2}M_kV_k'(M_k))\right\rangle + (\mathcal{N}+1)T \tag{5.46}$$

i.e.

$$2\frac{F}{T} - \frac{\partial F}{\partial T} - \frac{1}{T}\sum_i \epsilon_i\Gamma_i = \sum_{k=0}^{\mathcal{N}} \mathrm{Res}\,(V_k(M_k) - \frac{1}{2}M_kV_k'(M_k))W_k dz_k + (\mathcal{N}+1)T \tag{5.47}$$

which is equivalent to eq. (5.41).

Note that eq. (5.45), is nothing but the infinitesimal version of the rescaling $M_k \to \alpha M_k$. In particular one can choose $\alpha = \sqrt{T}$, and get directely from eq. (1.1):

$$F(g_{k,j},\tilde{g}_{k,j},\epsilon_i,T) = T^2F\left(g_{k,j}T^{j/2-1},\tilde{g}_{k,j}T^{j/2-1},\frac{\epsilon_i}{T},1\right) - \frac{\mathcal{N}+1}{2}T^2\ln T\,. \tag{5.48}$$

Taking the derivative of eq. (5.48) with respect to T, gives again eq. (5.41).

6. Other observables, leading order

6.1 Resolvents, leading order

So far, we have found that $W_0(z_0)$ and $W_{\mathcal{N}}(z_{\mathcal{N}})$ obey algebraic equations. We have been able to determine the resolvents for the matrices at the extremities of the chain, but not for intermediate matrices. We are going to determine $W_k(z_k)$ for $0 \le k \le \mathcal{N}$.

We start from eq. (5.23):

$$W_k(z) = -\frac{1}{2i\pi T}\oint_{\mathcal{C}_k}\frac{z_{k-1}}{z-z_k}dz_k \tag{6.1}$$

where the integration contour is $\mathcal{C}_k$ defined in 4.3. Indeed, the residue is a contour integral around $p_{\infty+}$, and since equation eq. (5.23) was derived formally for large z (i.e. order by order in the large z expansion), we have to assume that the contour of integration encloses all the r_k solutions of $z_k(p) = z$. Moreover, the residue is the sum of poles at the $p_{+j,k}(z_k)$'s and at $p_{\infty+}$:

$$\begin{aligned}
\frac{1}{2i\pi}\oint_{\mathcal{C}_k}\frac{z_{k-1}}{z_k-z}dz_k &= \sum_{j=1}^{r_k} z_{k-1}(p_{+j,k}(z_k)) + \frac{1}{2i\pi}\oint_{\mathcal{C}_k} z_{k-1}\frac{dz_k}{z_k} \\
&= \sum_{j=1}^{r_k} z_{k-1}(p_{+j,k}(z_k)) + \frac{1}{2i\pi}\oint_{\mathcal{C}_k} z_{k-1}\frac{V''(z_{k-1})dz_{k-1} - dz_{k-2}}{V_{k-1}'(z_{k-1}) - z_{k-2}} \\
&= \sum_{j=1}^{r_k} z_{k-1}(p_{+j,k}(z_k)) + \frac{1}{2i\pi}\oint_{\mathcal{C}_k}\frac{V''(z_{k-1})z_{k-1}}{V_{k-1}'(z_{k-1})}dz_{k-1}
\end{aligned}$$

$$= \sum_{j=1}^{r_k} z_{k-1}(p_{+j,k}(z_k)) + \frac{g_{k-1,d_{k-1}}}{g_{k-1,d_{k-1}+1}} \,. \tag{6.2}$$

Therefore:

$$TW_k(z) = \frac{g_{k-1,d_{k-1}}}{g_{k-1,d_{k-1}+1}} + \sum_{j=1}^{r_k} z_{k-1}(p_{+j,k}(z)) = \frac{g_{k+1,d_{k+1}}}{g_{k+1,d_{k+1}+1}} + \sum_{j=1}^{s_k} z_{k+1}(p_{-j,k}(z)) \,. \tag{6.3}$$

Notice that we have:

$$\sum_{j=-s_k}^{r_k} z_{k-1}(p_{j,k}(z)) = s_k V_k'(z) + \frac{g_{k+1,d_{k+1}}}{g_{k+1,d_{k+1}+1}} - \frac{g_{k-1,d_{k-1}}}{g_{k-1,d_{k-1}+1}} \,. \tag{6.4}$$

6.2 2-loop functions, leading order

From eq. (4.40) and eq. (6.3) We find ($l \leq k$):

$$\boxed{\left\langle \mathrm{Tr}\, \frac{1}{z_k - M_k}\, \mathrm{Tr}\, \frac{1}{z_l - M_l} \right\rangle_{\mathrm{c}} dz_k dz_l = - \sum_{i=1}^{r_k} \sum_{j=1}^{r_l} B(p_{+i,k}(z_k), p_{+j,l}(z_l))} \tag{6.5}$$

6.3 2-point one loop functions

6.3.1 $W_{k,k+1}$

Define the following polynomial of two variables:

$$\begin{aligned}
Q_{k,k+1}(z_k, z_{k+1}) &= \mathop{\mathrm{Res}}_{\infty} dz_0 \dots dz_{k-1} dz_{k+2} \dots dz_{\mathcal{N}} \frac{E(z_0, \dots, z_k, z_{k+1}, \dots, z_{\mathcal{N}})}{\prod_{j=0}^{\mathcal{N}} (V_j(z_j) - z_{j+1} - z_{j-1})} \\
&= E(\underline{Z}_0, \dots, \underline{Z}_{k-1}, z_k, z_{k+1}, \underline{Z}_{k+2}, \dots, \underline{Z}_{\mathcal{N}}) \\
&= (-1)^{r_k} \prod_{j=0}^{k-1} g_{j,d_j+1}^{r_j} \prod_{j=k+1}^{\mathcal{N}} g_{j,d_j+1}^{s_j} \prod_{0 \neq j = -s_k}^{r_k} (z_{k+1} - z_{k+1}(p_{j,k}(z_k))) \\
&= (-1)^{s_{k+1}} \prod_{j=0}^{k} g_{j,d_j+1}^{r_j} \prod_{j=k+2}^{\mathcal{N}} g_{j,d_j+1}^{s_j} \prod_{0 \neq j = -s_{k+1}}^{r_{k+1}} (z_k - z_k(p_{j,k+1}(z_{k+1})))
\end{aligned} \tag{6.6}$$

where the $\underline{Z}_j(z_k, z_{k+1})$ are defined by:

$$\begin{cases} \underline{Z}_k(z_k, z_{k+1}) = z_k \,, \quad \underline{Z}_{k+1}(z_k, z_{k+1}) = z_{k+1} \\ \underline{Z}_{j+1}(z_k, z_{k+1}) = V_l'(\underline{Z}_j(z_k, z_{k+1})) - \underline{Z}_{j-1}(z_k, z_{k+1}) \qquad j > k+1 \\ \underline{Z}_{j-1}(z_k, z_{k+1}) = V_l'(\underline{Z}_j(z_k, z_{k+1})) - \underline{Z}_{j+1}(z_k, z_{k+1}) \qquad j < k \end{cases} \tag{6.7}$$

We conjecture (proved in appendix B for $k = 0$):

$$\begin{aligned}
1 - W_{k,k+1}(z_k, z_{k+1}) &= \frac{Q_{k,k+1}(z_k, z_{k+1}) / \prod_{j=0}^{k-1} g_{j,d_j+1}^{r_j} \prod_{j=k+2}^{\mathcal{N}} g_{j,d_j+1}^{s_j}}{\prod_{j=1}^{r_k} (z_{k+1}(p_{+j,k}(z_k)) - z_{k+1}) \prod_{j=1}^{s_{k+1}} (z_k(p_{-j,k+1}(z_{k+1})) - z_k)} \\
&= g_{k,d_k+1}^{r_k} \frac{\prod_{j=1}^{r_{k+1}} (z_k - z_k(p_{+j,k+1}(z_{k+1})))}{\prod_{j=1}^{r_k} (z_{k+1}(p_{+j,k}(z_k)) - z_{k+1})} \\
&= g_{k+1,d_{k+1}+1}^{s_{k+1}} \frac{\prod_{j=1}^{s_k} (z_{k+1} - z_{k+1}(p_{-j,k}(z_k)))}{\prod_{j=1}^{s_{k+1}} (z_k(p_{-j,k+1}(z_{k+1})) - z_k)} \,.
\end{aligned} \tag{6.8}$$

Remark.

$$Q_{k,k+1}(z_k, z_{k+1}) = Q_{k-1,k}(V'_k(z_k) - z_{k+1}, z_k) \tag{6.9}$$

and we may conjecture that the spectral curves of the differential systems defined in the appendix of [5] are:

$$\det\left(z_{k+1}\mathbf{1}_{r_k+s_k} - \mathcal{D}_k(z_k)\right) = (-1)^{r_k}\,\frac{Q_{k,k+1}(z_k, z_{k+1})}{\prod_{j=0}^{k-1} g_{j,d_j+1}^{r_j}\prod_{j=k+1}^{\mathcal{N}} g_{j,d_j+1}^{s_j}} \tag{6.10}$$

so that the property eq. (6.9) would be nothing but the duality discovered in [5].

6.3.2 The function U in the large-N limit

Since the function U appears in the r.h.s. of the master loop equation, it is important to be able to compute it in the large-N limit.

Define (notice that $U_1 = U$):

$$U_k(z_0, z_k, \dots, z_{\mathcal{N}}) := \mathop{\mathrm{Pol}}_{z_1,\dots,z_{\mathcal{N}}}\; W(z_0, \dots, z_{\mathcal{N}})\, f_{k,\mathcal{N}}(z_k, \dots, z_{\mathcal{N}}) \prod_{j=1}^{k-1} z_j \tag{6.11}$$

We shall prove that:

$$\begin{aligned} U_k(z_0, z_k, \dots, z_{\mathcal{N}}) &= H_{k,\mathcal{N}}(z_k, \dots, z_{\mathcal{N}}) W_0(z_0) - \\ &\quad - \sum_{l=k}^{\mathcal{N}} \frac{P(Z_0, \dots, Z_{l-1}, z_l, \dots, z_{\mathcal{N}}) - P(Z_0, \dots, Z_l, z_{l+1}, \dots, z_{\mathcal{N}})}{z_l - Z_l} \times \\ &\quad \times H_{k,l-1}(z_k, \dots, z_{l-1}) \\ &= V'_{\mathcal{N}}(z_{\mathcal{N}}) - z_{\mathcal{N}-1} + \\ &\quad + \sum_{l=0}^{\mathcal{N}-1} \frac{E(Z_0, \dots, Z_{l-1}, z_l, \dots, z_{\mathcal{N}}) - E(Z_0, \dots, Z_l, z_{l+1}, \dots, z_{\mathcal{N}})}{z_l - Z_l} \times \\ &\quad \times H_{k,l-1}(z_k, \dots, z_{l-1}) \end{aligned} \tag{6.12}$$

where

$$H_{k,l}(z_k, \dots, z_l) := \mathrm{Pol}_{\,z_0,\dots,z_{\mathcal{N}}} \prod_{j=0}^{\mathcal{N}} \frac{1}{z_j - Z_j}\, f_{k,l}(z_k, \dots, z_l) \prod_{j<k} z_j \prod_{j>l} z_j \tag{6.13}$$

i.e. $H_{k,l}$ is a polynomial in $z_k, \dots, z_l$, and satisfies:

$$H_{k+1,k} = 1\,, \quad H_{k,l} = 0 \text{ if } k > l+1\,, \quad H_{k,l} = \frac{V'_k(z_k) - V'_k(Z_k)}{z_k - Z_k} H_{k+1,l} - H_{k+2,l}\,. \tag{6.14}$$

In particular for $k = 1$ eq. (6.12) reduces to:

$$\begin{aligned} U(z_0, \dots, z_{\mathcal{N}}) &= V'_{\mathcal{N}}(z_{\mathcal{N}}) - z_{\mathcal{N}-1} + \\ &\quad + \sum_{k=0}^{\mathcal{N}-1} \frac{E(Z_0, \dots, Z_k, z_{k+1}, \dots, z_{\mathcal{N}}) - E(Z_0, \dots, Z_{k+1}, z_{k+2}, \dots, z_{\mathcal{N}})}{z_{k+1} - Z_{k+1}} \times \end{aligned}$$

$$
\begin{aligned}
&\times H_{1,k}(z_1,\dots,z_k)\\
&= H_{1,\mathcal{N}}(z_1,\dots,z_{\mathcal{N}})W_0(z_0)-\\
&-\sum_{k=0}^{\mathcal{N}-1}\frac{P(Z_0,\dots,Z_k,z_{k+1},\dots,z_{\mathcal{N}})-P(Z_0,\dots,Z_{k+1},z_{k+2},\dots,z_{\mathcal{N}})}{z_{k+1}-Z_{k+1}}\times\\
&\times H_{1,k}(z_1,\dots,z_k)
\end{aligned}
\tag{6.15}
$$

Proof of eq. (6.12): let us define:

$$
A_k(z_0,z_k,\dots,z_{\mathcal{N}}) := \mathop{\mathrm{Pol}}_{z_1,\dots,z_{k-1},z_{k+1},\dots,z_{\mathcal{N}}}\prod_{j=1}^{k-1} z_j\, f_{k+1,\mathcal{N}}(z_{k+1},\dots,z_{\mathcal{N}})W(z_0,\dots,z_{\mathcal{N}}) \tag{6.16}
$$

$$
\begin{aligned}
S_k(z_0,z_k,\dots,z_{\mathcal{N}}) &:= \mathop{\mathrm{Pol}}_{z_0,\dots,z_{k-1},z_{k+1},\dots,z_{\mathcal{N}}} f_{0,k-1}(z_0,\dots,z_{k-1})\times\\
&\quad\times f_{k+1,\mathcal{N}}(z_{k+1},\dots,z_{\mathcal{N}})W(z_0,\dots,z_{\mathcal{N}})
\end{aligned}
\tag{6.17}
$$

$$
\begin{aligned}
T_k(z_0,z_k,\dots,z_{\mathcal{N}}) &:= \mathop{\mathrm{Pol}}_{z_0,\dots,z_{\mathcal{N}}} f_{0,k-1}(z_0,\dots,z_{k-1})V_k'(z_k)\times\\
&\quad\times f_{k+1,\mathcal{N}}(z_{k+1},\dots,z_{\mathcal{N}})W(z_0,\dots,z_{\mathcal{N}})\,.
\end{aligned}
\tag{6.18}
$$

From eq. (2.10) we have:

$$
U_k = \mathop{\mathrm{Pol}}_{z_k} V_k' A_k - U_{k+2} \tag{6.19}
$$

and from eq. (3.2), we have:

$$
(z_k - Z_k)A_k = U_{k+1} - S_k(Z_0,\dots,Z_{k-1},z_k,\dots,z_{\mathcal{N}})\,. \tag{6.20}
$$

That implies:

$$
\begin{aligned}
U_k &= \frac{V_k'(z_k)-V_k'(Z_k)}{z_k-Z_k}U_{k+1}-U_{k+2}-\\
&\quad-\frac{T_k(Z_0,\dots,Z_{k-1},z_k,\dots,z_{\mathcal{N}})-T_k(Z_0,\dots,Z_k,z_{k+1},\dots,z_{\mathcal{N}})}{z_k-Z_k}\,.
\end{aligned}
\tag{6.21}
$$

It is easy to prove (by recursion), that $f_{0,\mathcal{N}} - f_{0,k-1}V_k'(z_k)f_{k+1,\mathcal{N}}$ is linear in z_k, in other words we can write:

$$
f_{0,\mathcal{N}} = f_{0,k-1}V_k'(z_k)f_{k+1,\mathcal{N}} + z_k B_k \tag{6.22}
$$

where B_k is independent of z_k, therefore:

$$
T_k(z_0,\dots,z_k,\dots,z_{\mathcal{N}}) - P(z_0,\dots,z_k,\dots,z_{\mathcal{N}}) = \text{independent of } z_k\,. \tag{6.23}
$$

and thus eq. (6.21) can be rewritten:

$$
\begin{aligned}
U_k &= \frac{V_k'(z_k)-V_k'(Z_k)}{z_k-Z_k}U_{k+1}-U_{k+2}-\\
&\quad-\frac{P(Z_0,\dots,Z_{k-1},z_k,\dots,z_{\mathcal{N}})-P(Z_0,\dots,Z_k,z_{k+1},\dots,z_{\mathcal{N}})}{z_k-Z_k}\,.
\end{aligned}
\tag{6.24}
$$

From which, the initial conditions $U_{\mathcal{N}+1}=W_0$, $U_{\mathcal{N}+2}=0$ (easily derived) imply eq. (6.12).

6.3.3 The extremities correlator $W_{0,\mathcal{N}}$

From eq. (6.15) and eq. (3.14), we derive:

$$W_{0,\mathcal{N}}(z_0, z_{\mathcal{N}}) = \sum_{k=0}^{\mathcal{N}-1} \frac{E(Z_0, \dots, Z_k, \tilde{Z}_{k+1}, \dots, \tilde{Z}_{\mathcal{N}})}{(Z_k - \tilde{Z}_k)(Z_{k+1} - \tilde{Z}_{k+1})} \tag{6.25}$$

where:

$$Z_k := z_k(p_{+1,0}(z_0)) , \qquad \tilde{Z}_k := z_k(p_{-1,\mathcal{N}}(z_{\mathcal{N}})) . \tag{6.26}$$

7. Subleading epxansion

The aim of this section is to generalize the calculation of [10, 11], and compute the next to leading $1/N^2$ term in the topological expansion. In this purpose, we expand the various observables in a $1/N^2$ power series:

$$Z_k(z_0) = Z_k^{(0)}(z_0) + \frac{1}{N^2} Z_k^{(1)}(z_0) + \dots \tag{7.1}$$

$$P(z_0, \dots, z_{\mathcal{N}}) = P^{(0)}(z_0, \dots, z_{\mathcal{N}}) + \frac{1}{N^2} P^{(1)}(z_0, \dots, z_{\mathcal{N}}) + \dots \tag{7.2}$$

and so on. Then we expand eq. (3.1) to order $1/N^2$:

$$\sum_{k=1}^{\mathcal{N}} Z_k^{(1)}(z_0) E_k - P^{(1)}(z_0, Z_1^{(0)}, \dots, Z_{\mathcal{N}}^{(0)}) = U_{;0}^{(0)}(z_0, Z_1^{(0)}, \dots, Z_{\mathcal{N}}^{(0)}; z_0) \tag{7.3}$$

where:

$$E_k := \left. \frac{\partial E^{(0)}(z_0, \dots, z_{\mathcal{N}})}{\partial z_k} \right|_{z_j = Z_j^{(0)}} . \tag{7.4}$$

From the definition of the Z_k's eq. (2.8), we easily find:

$$Z_k^{(1)}(z_0) = \overline{H}_{1,k-1}\, Z_1^{(1)}(z_0) \tag{7.5}$$

where (the $H_{k,l}$'s have been defined in eq. (6.13)):

$$\overline{H}_{1,k} := H_{1,k}(Z_1^{(0)}, \dots, Z_k^{(0)}) \tag{7.6}$$

We have:

$$\overline{H}_{1,-1} = 0 , \qquad \overline{H}_{1,0} = 1 , \qquad \overline{H}_{1,k} = V_k''(Z_k^{(0)}) \overline{H}_{1,k-1} - \overline{H}_{1,k-2} \tag{7.7}$$

i.e.

$$\overline{H}_{1,k} = \left. \frac{\partial^k f_{1,k}(z_1, \dots, z_k)}{\partial z_1 \dots \partial z_k} \right|_{z_j = Z_j^{(0)}} . \tag{7.8}$$

Therefore eq. (7.2) reads:

$$Z_1^{(1)}(z_0) = \frac{P^{(1)}(z_0, Z_1^{(0)}, \dots, Z_{\mathcal{N}}^{(0)}) + U_{;0}^{(0)}(z_0, Z_1^{(0)}, \dots, Z_{\mathcal{N}}^{(0)}; z_0)}{\sum_{k=1}^{\mathcal{N}} \overline{H}_{1,k-1}\, E_k} . \tag{7.9}$$

Similarly to what was done in [10, 11], we determine the polynomial $P^{(1)}$, by the condition that $Z_1^{(1)}(z_0)$ has singularities only at the endpoints $e_{0,i}$ in the z_0–sheet. That condition is sufficient to determine all the unknown coefficients of $P^{(1)}$. Now, let us compute $U_{;0}^{(0)}$.

7.1 Computation of $U_{;0}(z_0,\dots,z_{\mathcal{N}};x)$

Notice that there is no explicit depedence on V_0 in eq. (2.23), therefore:

$$U_{;0}(z_0,\dots,z_{\mathcal{N}};z)=\frac{\partial U(z_0,\dots,z_{\mathcal{N}})}{\partial V_0(z)}\,. \tag{7.10}$$

For $k\geq 1$, write $z_k=Z_k^{(0)}+\zeta_k$, and Taylor expand eq. (6.15) in the ζ's:

$$\begin{aligned}U(z_0,\dots,z_{\mathcal{N}}) &= V'_{\mathcal{N}}(z_{\mathcal{N}})-z_{\mathcal{N}-1}-\\ &-\sum_{k=1}^{\mathcal{N}}\left(E_k+\frac{1}{2}\zeta_k E_{k,k}+\sum_{i=k+1}^{\mathcal{N}}\zeta_i E_{k,i}\right)\left(\overline{H}_{1,k-1}+\sum_{i=1}^{k-1}\zeta_i\overline{H}_{1,k-1;i}\right)\end{aligned} \tag{7.11}$$

where E_k was defined in eq. (7.4), and $E_{k,l}$ is:

$$E_{k,l}:=\left.\frac{\partial^2 E^{(0)}(z_0,\dots,z_{\mathcal{N}})}{\partial z_k\partial z_l}\right|_{z_j=Z_j^{(0)}} \tag{7.12}$$

$\overline{H}_{1,k}$ was defined in eq. (7.6) and:

$$\overline{H}_{1,k;i}:=\left.\frac{\partial H_{1,k}(z_1,\dots,z_k)}{\partial z_i}\right|_{z_j=Z_j^{(0)}}\,. \tag{7.13}$$

From eq. (6.13) and eq. (7.6) we derive:

$$\overline{H}_{1,k;i}=\frac{1}{2}V_i'''(Z_i^{(0)})\,\overline{H}_{1,i-1}\,\overline{H}_{i+1,k} \tag{7.14}$$

Then take the $\partial/\partial V_0(z)$ derivative, using $\partial\zeta_k/\partial V_0=-\partial Z_k/\partial V_0$, and take the $\zeta\to 0$ limit:

$$\begin{aligned}U_{;0}^{(0)}(Z_0,Z_1^{(0)},\dots,Z_{\mathcal{N}}^{(0)};z) &= \sum_{k=1}^{\mathcal{N}}\left(\frac{\partial E_k}{\partial V_0(z)}-\frac{\partial Z_k^{(0)}}{\partial V_0(z)}\frac{E_{k,k}}{2}-\sum_{j>k}\frac{\partial Z_j^{(0)}}{\partial V_0(z)}E_{k,j}\right)\overline{H}_{1,k-1}+\\ &+\sum_{k=1}^{\mathcal{N}}E_k\left(\frac{\partial\overline{H}_{1,k-1}}{\partial V_0(z)}-\sum_{j=1}^{k-1}\frac{\partial Z_j^{(0)}}{\partial V_0(z)}\overline{H}_{1,k-1;j}\right)\,.\end{aligned} \tag{7.15}$$

From eq. (2.8), one easily derives:

$$\frac{\partial Z_{k+1}^{(0)}}{\partial V_0(z)}=\frac{\partial Z_k^{(0)}}{\partial V_0(z)}V_k''(Z_k^{(0)})-\frac{\partial Z_{k-1}^{(0)}}{\partial V_0(z)}=\overline{H}_{1,k}\frac{\partial Z_1^{(0)}}{\partial V_0(z)} \tag{7.16}$$

and from eq. (7.6)

$$\frac{\partial\overline{H}_{1,k-1}}{\partial V_0(z)}=\sum_{j=1}^{k-1}\frac{\partial Z_j^{(0)}}{\partial V_0(z)}V_j'''(Z_j^{(0)})\overline{H}_{1,j-1}\overline{H}_{j+1,k-1}=2\sum_{j=1}^{k-1}\frac{\partial Z_j^{(0)}}{\partial V_0(z)}\overline{H}_{1,k-1;j}\,. \tag{7.17}$$

7.2 Computation of $\partial E_k/\partial V_0(z)$

Let us define $\overline{Z}_j(z_0,z_1)$:

$$\overline{Z}_0(z_0,z_1) := z_0\,,\qquad \overline{Z}_1(z_0,z_1) := z_1\,,\qquad \overline{Z}_{j+1}(z_0,z_1) = V_j'(\overline{Z}_j(z_0,z_1)) - \overline{Z}_{j-1}(z_0,z_1)\,. \tag{7.18}$$

We have:

$$\left.\frac{\partial \overline{Z}_j(z_0,z_1)}{\partial z_1}\right|_{z_1=Z_1^{(0)}} = \overline{H}_{1,j-1}\,,\qquad \left.\frac{\partial^2 \overline{Z}_k(z_0,z_1)}{\partial z_1^2}\right|_{z_1=Z_1^{(0)}} = \sum_{j=1}^{k-1} V_j'''(Z_j^{(0)})\,\overline{H}_{1,j-1}^2\,\overline{H}_{j+1,k-1}\,. \tag{7.19}$$

Consider the polynomials defined in eq. (6.6):

$$\begin{aligned} Q_{0,1}(z_0,z_1) &:= E^{(0)}(\overline{Z}_0(z_0,z_1),\ldots,\overline{Z}_{\mathcal{N}}(z_0,z_1)) \\ &= C(z_1 - Z_1^{(0)}(z_0))\prod_{j=1}^{s_0}(z_1 - z_1(p_{-j,0}(z_0)))) \end{aligned} \tag{7.20}$$

(where $C = -\prod_{j=1}^{\mathcal{N}} g_{j,d_j+1}^{s_j}$), and take its derivative (using eq. (7.19)):

$$\left.\frac{\partial Q_{0,1}(z_0,z_1)}{\partial z_1}\right|_{z_1=Z_1^{(0)}} = \sum_{j=1}^{\mathcal{N}} E_j\,\overline{H}_{1,j-1} = C\prod_{j=1}^{s_0}(Z_1^{(0)} - z_1(p_{-j,0}(z_0)))\,. \tag{7.21}$$

That implies:

$$\begin{aligned} &\sum_{j=1}^{\mathcal{N}} \frac{\partial E_j}{\partial V_0(z)}\,\overline{H}_{1,j-1} + \sum_{j=1}^{\mathcal{N}} E_j\,\frac{\partial \overline{H}_{1,j-1}}{\partial V_0(z)} = \\ &\quad = \sum_{l=1}^{s_0}\left(\frac{\partial Z_1^{(0)}}{\partial V_0(z)} - \left.\frac{\partial z_1(p_{-j,0}(z_0))}{\partial V_0(z)}\right|_{z_0}\right)\frac{\sum_{j=1}^{\mathcal{N}} E_j\,\overline{H}_{1,j-1}}{Z_1^{(0)} - z_1(p_{-l,0}(z_0))} \end{aligned} \tag{7.22}$$

Plugging that into eq. (7.15), we get:

$$\begin{aligned} U_{;0}^{(0)}(Z_0,Z_1^{(0)},\ldots,Z_{\mathcal{N}}^{(0)};z) &= \sum_{l=1}^{s_0}\left(\frac{\partial Z_1^{(0)}}{\partial V_0(z)} - \left.\frac{\partial z_1(p_{-j,0}(z_0))}{\partial V_0(z)}\right|_{z_0}\right)\frac{\sum_{j=1}^{\mathcal{N}} E_j\,\overline{H}_{1,j-1}}{Z_1^{(0)} - z_1(p_{-l,0}(z_0))} - \\ &\quad - \sum_{k=1}^{\mathcal{N}} \frac{\partial Z_k^{(0)}}{\partial V_0(z)}\frac{E_{k,k}}{2}\,\overline{H}_{1,k-1} - \sum_{k=1}^{\mathcal{N}}\sum_{j>k}\frac{\partial Z_j^{(0)}}{\partial V_0(z)} E_{k,j}\,\overline{H}_{1,k-1} - \\ &\quad - \sum_{k=1}^{\mathcal{N}}\sum_{j=1}^{k-1}\frac{\partial Z_j^{(0)}}{\partial V_0(z)} E_k\,\overline{H}_{1,k-1;j} \\ &= \sum_{l=1}^{s_0}\left(\frac{\partial Z_1^{(0)}}{\partial V_0(z)} - \left.\frac{\partial z_1(p_{-j,0}(z_0))}{\partial V_0(z)}\right|_{z_0}\right)\frac{\sum_{j=1}^{\mathcal{N}} E_j\,\overline{H}_{1,j-1}}{Z_1^{(0)} - z_1(p_{-l,0}(z_0))} - \\ &\quad - \frac{\partial Z_1^{(0)}}{\partial V_0(z)}\sum_{k=1}^{\mathcal{N}}\left(\frac{E_{k,k}}{2}\,\overline{H}_{1,k-1}^2 + \sum_{j>k} E_{k,j}\,\overline{H}_{1,k-1}\,\overline{H}_{1,j-1}\right) - \end{aligned}$$

$$
-\frac{1}{2}\frac{\partial Z_1^{(0)}}{\partial V_0(z)}\sum_{k=1}^{\mathcal{N}}\sum_{j=1}^{k-1} E_k\, V_j'''(Z_j^{(0)})\,\overline{H}_{1,j-1}^2\,\overline{H}_{j+1,k-1}
$$

$$
\begin{aligned}
&= -\sum_{l=1}^{s_0}\left.\frac{\partial z_1(p_{-j,0}(z_0)}{\partial V_0(z))}\right|_{z_0}\frac{\left(\sum_{j=1}^{\mathcal{N}} E_j\,\overline{H}_{1,j-1}\right)}{Z_1^{(0)}-z_1(p_{-l,0}(z_0))}+\\
&\quad +\frac{\partial Z_1^{(0)}}{\partial V_0(z)}\sum_{l=1}^{s_0}\frac{\left(\sum_{j=1}^{\mathcal{N}} E_j\,\overline{H}_{1,j-1}\right)}{Z_1^{(0)}-z_1(p_{-l,0}(z_0))}-\\
&\quad -\frac{1}{2}\frac{\partial Z_1^{(0)}}{\partial V_0(z)}\sum_{k=1}^{\mathcal{N}}\sum_{j=1}^{\mathcal{N}} E_{k,j}\,\overline{H}_{1,k-1}\,\overline{H}_{1,j-1}-\\
&\quad -\frac{1}{2}\frac{\partial Z_1^{(0)}}{\partial V_0(z)}\sum_{k=1}^{\mathcal{N}}\sum_{j=1}^{k-1} E_k\, V_j'''(Z_j^{(0)})\,\overline{H}_{1,j-1}^2\,\overline{H}_{j+1,k-1}\,. \qquad (7.23)
\end{aligned}
$$

Take the second derivative of eq. (7.20):

$$
\begin{aligned}
\left.\frac{\partial^2 Q_{0,1}(z_0,z_1)}{\partial z_1^2}\right|_{z_1=Z_1^{(0)}} &= 2\left(\sum_{j=1}^{\mathcal{N}} E_j\,\overline{H}_{1,j-1}\right)\sum_{l=1}^{s_0}\frac{1}{Z_1^{(0)}-z_1(p_{-l,0}(z_0))}\\
&= \sum_{k=1}^{\mathcal{N}}\sum_{j=1}^{\mathcal{N}} E_{k,j}\,\overline{H}_{1,j-1}\,\overline{H}_{1,j-1}+\sum_{k=1}^{\mathcal{N}} E_k\left.\frac{\partial^2 \overline{Z}_k(z_0,z_1)}{\partial z_1^2}\right|_{z_1=Z_1} \qquad (7.24)
\end{aligned}
$$

plugging eq. (7.24) and eq. (7.19) into eq. (7.23), we get:

$$
\boxed{\frac{U_{;0}^{(0)}(Z_0,Z_1^{(0)},\dots,Z_{\mathcal{N}}^{(0)};z)}{\sum_{j=1}^{\mathcal{N}} E_j\,\overline{H}_{1,j-1}} = -\sum_{l=1}^{s_0}\frac{1}{Z_1^{(0)}-z_1(p_{-l,0}(z_0))}\left.\frac{\partial z_1(p_{-j,0}(z_0))}{\partial V_0(z)}\right|_{z_0}} \qquad (7.25)
$$

which can be compared to [11].

7.3 Next to leading order

Using eq. (7.25) into eq. (7.9), and using eq. (4.40), we get the $1/N^2$ correction for the resolvent:

$$
\begin{aligned}
Z_1^{(1)}(z_0) &= \frac{P^{(1)}(z_0,Z_1^{(0)},\dots,Z_{\mathcal{N}}^{(0)})}{\sum_{k=1}^{\mathcal{N}}\overline{H}_{1,k-1}\,E_k}-\sum_{l=1}^{s_0}\frac{1}{Z_1^{(0)}-z_1(p_{-l,0}(z_0))}\left.\frac{\partial z_1(p_{-j,0}(z_0))}{\partial V_0(z_0)}\right|_{z_0}\\
&= \sum_{l=1}^{s_0}\frac{1}{z_1(p_{1,0}(z_0))-z_1(p_{-l,0}(z_0))}\,\frac{B(p_{-l,0}(z_0),p_{1,0}(z_0))}{dz_0(p_{1,0}(z_0))dz_0(p_{-l,0}(z_0))}+\\
&\quad +\frac{P^{(1)}(z_0,Z_1^{(0)},\dots,Z_{\mathcal{N}}^{(0)})}{\sum_{k=1}^{\mathcal{N}}\overline{H}_{1,k-1}\,E_k} \qquad (7.26)
\end{aligned}
$$

and we remind that $P^{(1)}$ is completely determined by the condition that $Z_1^{(1)}(z_0)$ has singularities only at the endpoints $e_{0,i}$ in the z_0–sheet, and has vanishing $\mathcal{B}$–cylce integrals.

From there, it should be possible to extend the calculation of [10, 11] to the chain of matrices, and compute the $1/N^2$ term in the free energy for the chain of matrices. This will be left for a later work.

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8. Examples

8.1 Example: one-cut asumption (genus zero)

Let us assume that the genus is $g = 0$.

It was already discussed in [10] that for multimatrix models, the so called one-cut asumption should be replaced by a genus zero asumption. Indeed, in that case, the number of z_0–endpoints, i.e. the number of zeroes of dz_0 is equal to the number of sheets, therefore there is exactly one cut in the physical sheet. $\mathcal{E}$ is in one to one correspondance with the complex plane $\mathbf{C}$, and it is well known [16, 17] that the parametrization eq. (4.34) is rational, and with $s = \frac{p-p_{\infty-}}{p-p_{\infty+}}$, it can be written:

$$z_k(s) = \sum_{i=-s_k}^{r_k} \alpha_{k,i}\, s^i\,. \tag{8.1}$$

$z_k(s)$ has two poles: one pole of degree r_k at $s = \infty$, and one pole of degree s_k at $s = 0$. This parametrization is identical to the one found in [13] by the biorthogonal polynomial's method.

The set of equations:

$$\begin{cases} V_k'(z_k(s)) = z_{k+1}(s) + z_{k-1}(s) \\ V_0'(z_0(s)) - z_1(s) \underset{s\to\infty}{\sim} \dfrac{T}{\alpha_{0,1}s} \\ V_{\mathcal{N}}'(z_{\mathcal{N}}(s)) - z_{\mathcal{N}-1}(s) \underset{s\to 0}{\sim} \dfrac{Ts}{\alpha_{\mathcal{N},-1}} \\ \alpha_{0,1} = \alpha_{\mathcal{N},-1} = \gamma = \tilde{\gamma} \end{cases} \tag{8.2}$$

is sufficient to determine all the $\alpha_{k,i}$ (one is free to impose $\alpha_{0,1} = \alpha_{\mathcal{N},-1}$ because s can be changed into any constant times s).

The abelian differential of the third kind is:

$$dS = -\frac{ds}{s}\,. \tag{8.3}$$

The Bergmann kernel is:

$$B(s, s') = \frac{ds\, ds'}{(s-s')^2}\,. \tag{8.4}$$

8.2 Example: gaussian case

Consider all potentials quadratic (i.e. $d_k = 1$):

$$V_k(z_k) := \frac{g_k}{2} z_k^2\,. \tag{8.5}$$

A direct computation of the matrix integral eq. (1.2) gives the free energy:

$$F = F^{(0)} = \frac{T^2}{2} \ln D_{0,\mathcal{N}} - \frac{\mathcal{N}+1}{2} T^2 \ln T\,. \tag{8.6}$$

Define:

$$D_{k,l} := \det \begin{pmatrix} g_k & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & g_l \end{pmatrix}, \qquad D_{k,k-1} := 1\,, \qquad D_{k,k-2} := 0\,. \tag{8.7}$$

The function $f_{k,l}$ of eq. (2.10) is:

$$f_{k,l} = z_k \dots z_l\, D_{k,l} \tag{8.8}$$

thus the polynomial P is a constant:

$$P(z_0, \dots, z_{\mathcal{N}}) = D_{0,\mathcal{N}} \tag{8.9}$$

and the polynomial E is:

$$E(z_0, \dots, z_{\mathcal{N}}) = (g_0 z_0 - z_1)(g_{\mathcal{N}} z_{\mathcal{N}} - z_{\mathcal{N}-1}) - TP\,. \tag{8.10}$$

The leading order loop equations are thus:

$$z_{k+1} + z_{k-1} = g_k z_k\,, \qquad z_{-1} z_{\mathcal{N}+1} = TP\,. \tag{8.11}$$

Note that eq. (4.15) implies that the genus is necessarily $g = 0$, and thus there is a rational parametrization of the form eq. (8.1), with $r_k = s_k = 1$, namely:

$$z_k = \sqrt{\frac{T}{D_{0,\mathcal{N}}}}\left(D_{0,k-1} s + D_{k+1,\mathcal{N}} s^{-1}\right). \tag{8.12}$$

We find:

$$\gamma = \tilde{\gamma} = \sqrt{\frac{T}{D_{0,\mathcal{N}}}} \tag{8.13}$$

and:

$$\mu = T + T \ln D_{0,\mathcal{N}} - T \ln T = T - T \ln \gamma\tilde{\gamma} \tag{8.14}$$

and it is easy to check that eq. (5.2) coincides with eq. (8.6).

8.3 Loop equations of the 2 matrix model

Let us take $T = 1$. The 2-matrix model corresponds to $\mathcal{N} = 1$ we have:

$$f_{0,1}(z_0, z_1) = V_0'(z_0) V_1'(z_1) - z_0 z_1\,, \quad f_{1,1}(z_1) = V_1'(z_1) \tag{8.15}$$

i.e.

$$P(z_0, z_1) = \frac{1}{N}\left\langle \operatorname{tr} \frac{V_0'(z_0) - V_0'(M_0)}{z_0 - M_0} \frac{V_1'(z_1) - V_1'(M_1)}{z_1 - M_1} \right\rangle - 1 \tag{8.16}$$

$$U(z_0, z_1; z) = \left\langle \operatorname{tr} \frac{1}{z_0 - M_0} \frac{V_1'(z_1) - V_1'(M_1)}{z_1 - M_1} \operatorname{tr} \frac{1}{z - M_0} \right\rangle_{\mathrm{c}} \tag{8.17}$$

and the master loop equation reads:

$$(V_0'(Z_0) - Z_1)(V_1'(Z_1) - Z_0) - P(Z_0, Z_1) = \frac{1}{N^2} U(Z_0, Z_1; Z_0) \tag{8.18}$$

with $Z_1 = V_0'(z_0) - W_0(z_0)$. We recover the equation of [12, 13, 10].

8.4 The 3 matrix model

Let us take $T = 1$. The 3-matrix model corresponds to $\mathcal{N} = 2$, i.e.:

$$f_{0,2}(z_0, z_1, z_2) = V_0'(z_0)V_1'(z_1)V_2'(z_2) - z_0 z_1 V_2'(z_2) - z_2 z_1 V_0'(z_0) \tag{8.19}$$
$$f_{1,2}(z_1, z_2) = V_1'(z_1)V_2'(z_2) - z_1 z_2 \tag{8.20}$$

i.e.

$$\begin{aligned} P(z_0, z_1, z_2) = & \frac{1}{2N}\left\langle \mathrm{tr}\, \frac{V_0'(z_0) - V_0'(M_0)}{z_0 - M_0} \frac{V_1'(z_1) - V_1'(M_1)}{z_1 - M_1} \frac{V_2'(z_2) - V_2'(M_2)}{z_2 - M_2} \right\rangle + \\ & + \frac{1}{2N}\left\langle \mathrm{tr}\, \frac{V_2'(z_2) - V_2'(M_2)}{z_2 - M_2} \frac{V_1'(z_1) - V_1'(M_1)}{z_1 - M_1} \frac{V_0'(z_0) - V_0'(M_0)}{z_0 - M_0} \right\rangle - \\ & - \frac{1}{N}\left\langle \mathrm{tr}\, \frac{V_0'(z_0) - V_0'(M_0)}{z_0 - M_0} \right\rangle - \frac{1}{N}\left\langle \mathrm{tr}\, \frac{V_2'(z_2) - V_2'(M_2)}{z_2 - M_2} \right\rangle \end{aligned} \tag{8.21}$$

$$\begin{aligned} U(z_0, z_1, z_2; z) = & \frac{1}{2}\left\langle \mathrm{tr}\, \frac{1}{z_0 - M_0} \frac{V_1'(z_1) - V_1'(M_1)}{z_1 - M_1} \frac{V_2'(z_2) - V_2'(M_2)}{z_2 - M_2}\, \mathrm{tr}\, \frac{1}{z - M_0} \right\rangle_{\mathrm{c}} + \\ & + \frac{1}{2}\left\langle \mathrm{tr}\, \frac{V_2'(z_2) - V_2'(M_2)}{z_2 - M_2} \frac{V_1'(z_1) - V_1'(M_1)}{z_1 - M_1} \frac{1}{z_0 - M_0}\, \mathrm{tr}\, \frac{1}{z - M_0} \right\rangle_{\mathrm{c}} - \\ & - \left\langle \mathrm{tr}\, \frac{1}{z_0 - M_0}\, \mathrm{tr}\, \frac{1}{z - M_0} \right\rangle_{\mathrm{c}} \end{aligned} \tag{8.22}$$

and the master loop equation is:

$$(V_0'(Z_0) - Z_1)(V_2'(Z_2) - Z_1) - P(Z_0, Z_1, Z_2) = \frac{1}{N^2} U(Z_0, Z_1, Z_2; Z_0) \tag{8.23}$$

with $Z_0 = z_0$, $Z_1(z_0) = V_0'(z_0) - W_0(z_0)$ and $Z_2(z_0) = V_1'(Z_1) - Z_0$.

Large N one loop functions. We define:

$$\tilde{Z}_1(z_2) = V_2'(z_2) - W_2(z_2)\,, \qquad \tilde{Z}_0(z_2) = V_1'(\tilde{Z}_1(z_2)) - z_2 \tag{8.24}$$
$$W_{\hat{0},1}(z_0, z_1) = \mathop{\mathrm{Pol}}_{z_0} V_0'(z_0) W_{0,1}(z_0, z_1) \tag{8.25}$$
$$W_{1,\hat{2}}(z_1, z_2) = \mathop{\mathrm{Pol}}_{z_2} V_2'(z_2) W_{1,2}(z_1, z_2) \tag{8.26}$$
$$W_{\hat{0},1,\hat{2}}(z_0, z_1, z_2) = \mathop{\mathrm{Pol}}_{z_0, z_2} V_0'(z_0) V_2'(z_2) W(z_0, z_1, z_2)\,. \tag{8.27}$$

equ. (6.25) reads:

$$W_{0,2}(z_0, z_2) = \frac{1}{Z_1 - \tilde{Z}_1}\left(\frac{E(z_0, \tilde{Z}_1, z_2)}{z_0 - \tilde{Z}_0} - \frac{E(z_0, Z_1, z_2)}{z_2 - Z_2} \right) \tag{8.28}$$

eq. (6.12) reads:

$$U_2(z_0, z_2) = \mathop{\mathrm{Pol}}_{z_2} V_2'(z_2) W_{0,2}(z_0, z_2) = \frac{E(z_0, Z_1, z_2)}{z_2 - Z_2} \tag{8.29}$$

eq. (6.15) reads:

$$\begin{aligned} U(z_0,z_1,z_2) &= P_{1,2}(z_0,z_1,z_2) \\ &= V_2'(z_2) - z_1 + \frac{E(z_0,z_1,z_2)-E(z_0,Z_1,z_2)}{z_1-Z_1} + \\ &\quad + \frac{V_1'(z_1)-V_1'(Z_1)}{z_1-Z_1}\frac{E(z_0,Z_1,z_2)}{z_2-Z_2} \end{aligned} \tag{8.30}$$

eq. (3.5) implies:

$$\begin{aligned} P_{1,1} &= \frac{V_1'(z_1)-V_1'(Z_1)}{z_1-Z_1} W_{0,2}(z_0,z_2) - \frac{P_{1,2}(z_0,z_1,z_2)-P_{1,2}(z_0,Z_1,z_2)}{z_1-Z_1} \\ &= \frac{(V_1'-z_0-z_2)}{(Z_1-\tilde{Z}_1)}\left(\frac{E(z_0,\tilde{Z}_1,z_2)}{(z_1-\tilde{Z}_1)(z_0-\tilde{Z}_0)} - \frac{E(z_0,Z_1,z_2)}{(z_1-Z_1)(z_2-Z_2)}\right) + \\ &\quad +1 - \frac{E(z_0,z_1,z_2)}{(z_1-Z_1)(z_1-\tilde{Z}_1)} \end{aligned} \tag{8.31}$$

We have the relationships:

$$(z_1-\tilde{Z}_1)P_{0,0}(z_0,z_1,z_2) = \frac{E(z_0,\tilde{Z}_1,z_2)}{z_0-\tilde{Z}_0} - W_{\hat{0},1,\hat{2}}(z_0,z_1,z_2) \tag{8.32}$$

$$\begin{aligned} (Z_1-\tilde{Z}_1)W(z_0,z_1,z_2) &= \frac{E(z_0,\tilde{Z}_1,z_2)}{(z_1-\tilde{Z}_1)(z_0-\tilde{Z}_0)} - \frac{E(z_0,Z_1,z_2)}{(z_1-Z_1)(z_2-Z_2)} + \\ &\quad + \frac{(Z_1-\tilde{Z}_1)W_{\hat{0},1,\hat{2}}(z_0,z_1,z_2)}{(z_1-Z_1)(z_1-\tilde{Z}_1)}. \end{aligned} \tag{8.33}$$

But so far, we have not been able to compute $W_{\hat{0},1,\hat{2}}$. We conjecture:

$$\begin{aligned} (z_0+z_2-V_1'(z_1))W_{\hat{0},1,\hat{2}}(z_0,z_1,z_2) &= E(z_0,z_1,z_2) - (V_0'(z_0)-z_1-W_{\hat{0},1}(z_0,z_1)) \times \\ &\quad \times (V_2'(z_2)-z_1-W_{1,\hat{2}}(z_1,z_2)) \end{aligned} \tag{8.34}$$

indeed, both sides are polynomials in z_0 and z_2 with the same degree and same large z_0, z_1 and z_2 behaviours.

9. Conclusion

We have written the loop equations in a very explicit way. To leading order the loop equations become algebraic, and using that fact, we have derived many observables, like the free energy and many correlators.

We have computed some mixed correlators and we have given conjecture for others. So far, the complete one–loop function has not been computed, and we don't even have a conjecture (except in the 3-matrix model). The mixed correlators are important in the study of boundary operators [20].

It would also be interesting to generalize [10] to the chain of matrices, and find the next to leading large-N corrections to the free energy.

Acknowledgments

The author would like to thank V. Kazakov for encouraging him to translate in english (and add new results to) old notes written in french, and to thank M. Bertola for fruitfull discussions and careful reading. When this work was in progress, M. Bertola found independent proofs for some of the formulae in section 5, which will be published later.

A. Some usefull formulas with residues and polynomial parts

We make an abundant use of the following formulas:

$$
\begin{aligned}
1 &= \mathop{\rm Pol}_z \frac{z}{z-Z} = \mathop{\rm Pol}_z \frac{z}{z-M} = -\mathop{\rm Res} \frac{dz}{z-Z} \\
&= -\mathop{\rm Res} \frac{zdz}{(z-Z)(z-M)}
\end{aligned} \tag{A.1}
$$

$$
\mathop{\rm Pol}_z \frac{V'(z)}{z-Z} = \frac{V'(z)-V'(Z)}{z-Z}, \qquad \mathop{\rm Res} \frac{V'(z)dz}{z-Z} = -V'(Z) \tag{A.2}
$$

$$
\mathop{\rm Pol}_z \frac{V'(z)}{(z-Z)(z-M)} = \frac{1}{z-Z}\left(\frac{V'(z)-V'(M)}{z-M} - \frac{V'(Z)-V'(M)}{Z-M}\right) \tag{A.3}
$$

$$
\mathop{\rm Res} \frac{V'(z)dz}{(z-Z)(z-M)} = -\frac{V'(Z)-V'(M)}{Z-M} \tag{A.4}
$$

$$
\frac{V'(M)}{z-M} = \frac{V'(z)}{z-M} - \frac{V'(z)-V'(M)}{z-M} = \frac{V'(z)}{z-M} - \mathop{\rm Pol}_z \frac{V'(z)}{z-M} \tag{A.5}
$$

$$
\frac{M}{z-M} = \frac{z}{z-M} - 1 = \frac{z}{z-M} - \mathop{\rm Pol}_z \frac{z}{z-M}\,. \tag{A.6}
$$

B. Determination of $W_{0,1}$

Define the following polynomial of z_0:

$$
W_{\hat{0},1}(z_0,z_1) = \mathop{\rm Pol}_{z_0} V_0'(z_0) W_{0,1}(z_0,z_1) \tag{B.1}
$$

eq. (3.5) implies the following identity (to large-N leading order):

$$
(z_1 - Z_1)(1 - W_{0,1}(z_0,z_1)) = z_1 - V_0'(z_0) + W_{\hat{0},1}(z_0,z_1) \tag{B.2}
$$

which can also be written:

$$
\left(1 - \frac{Z_1}{z_1}\right)(1 - W_{0,1}(z_0,z_1)) = 1 - \frac{V_0'(z_0) - W_{\hat{0},1}(z_0,z_1)}{z_1}\,. \tag{B.3}
$$

Notice that the r.h.s. is a polynomial in z_0.

Take the log:

$$
\ln\left(1 - \frac{Z_1}{z_1}\right) = -\ln\left(1 - W_{0,1}(z_0,z_1)\right) + \ln\left(1 - \frac{V_0'(z_0) - W_{\hat{0},1}(z_0,z_1)}{z_1}\right) \tag{B.4}
$$

take the fractionary part ($\mathrm{Frac}\, f := f - \mathrm{Pol}\, f$), we find:

$$-\ln(1 - W_{0,1}(z_0,z_1)) = \mathop{\mathrm{Frac}}_{z_0}\left(\ln(1 - \frac{Z_1(z_0)}{z_1})\right) \tag{B.5}$$

i.e.

$$W_{0,1}(z_0,z_1) = 1 - \exp\left(-\mathop{\mathrm{Frac}}_{z_0}\left(\ln(1 - \frac{Z_1(z_0)}{z_1})\right)\right) \tag{B.6}$$

in other words:

$$\begin{aligned} -\ln(1 - W_{0,1}(z_0,z_1)) &= \frac{1}{2i\pi}\oint_{p\in\mathcal{C}_0} \frac{dZ_0(p)}{z_0 - Z_0(p)} \ln\left(1 - \frac{Z_1(p)}{z_1}\right) \\ &= \frac{1}{2i\pi}\oint_{\mathcal{C}_1} \frac{dZ_1(p)}{Z_1(p) - z_1} \ln\left(1 - \frac{Z_0(p)}{z_0}\right) \\ &= \mathop{\mathrm{Frac}}_{z_1}\left(\ln\left(1 - \frac{Z_0(z_1)}{z_0}\right)\right) \\ &= \ln(1 - \frac{Z_0(z_1)}{z_0}) - \mathop{\mathrm{Pol}}_{z_1}\left(\ln\left(1 - \frac{Z_0(z_1)}{z_0}\right)\right) \end{aligned} \tag{B.7}$$

where the second equality is obtained by integration by parts.

In other words we have that:

$$(1 - W_{0,1}(z_0,z_1))(z_1 - Z_1(z_0)) = \text{Polynomial in } z_0 \tag{B.8}$$

$$(1 - W_{0,1}(z_0,z_1))\prod_{j=1}^{s_1}(z_0 - Z_0(p_{-j,1}(z_1))) = \text{Polynomial in } z_1 \tag{B.9}$$

that implies that:

$$1 - W_{0,1}(z_0,z_1) = \frac{Q_{0,1}(z_0,z_1)}{(z_1 - Z_1(z_0))\prod_{j=1}^{s_1}(z_0 - Z_0(p_{-j,1}(z_1)))} \tag{B.10}$$

where $Q_{0,1}(z_0,z_1)$ is some polynomial in both variables, of degree $r_1 + s_1, r_0 + s_0$, which vanishes each time there exists p such that $z_0 = Z_0(p)$ and $z_1 = Z_1(p)$. That implies that:

$$\begin{aligned} Q_{0,1}(z_0,z_1) &\propto \prod_{j=1}^{r_1}(z_0 - Z_0(p_{+j,1}(z_1)))\prod_{j=1}^{s_1}(z_0 - Z_0(p_{-j,1}(z_1))) \\ &\propto \prod_{j=1}^{r_0}(z_1 - Z_1(p_{+j,0}(z_0)))\prod_{j=1}^{s_0}(z_1 - Z_1(p_{-j,0}(z_0))) \end{aligned} \tag{B.11}$$

i.e. $Q_{0,1}$ must be equal to (same degree, same zeroes):

$$Q_{0,1}(z_0,z_1) \propto \oint dz_2 \ldots dz_{\mathcal{N}} \frac{E(z_0,z_1,z_2,\ldots,z_{\mathcal{N}})}{\prod_{j=1}^{\mathcal{N}-1}(z_{j+1} + z_{j-1} - V_j(z_j))}. \tag{B.12}$$

We thus have:

$$\boxed{W_{0,1}(z_0,z_1) = 1 - \frac{Q_{0,1}(z_0,z_1)}{(z_1 - Z_1(z_0))\prod_{j=1}^{s_1}(z_0 - Z_0(p_{-j,1}(z_1)))}} \tag{B.13}$$

i.e.

$$\boxed{W_{0,1}(z_0,z_1) = 1 + g_{0,d_0+1}\,\frac{\prod_{j=1}^{r_1}(z_0 - Z_0(p_{+j,1}(z_1)))}{(z_1 - Z_1(p_{+1,0}(z_0)))}} \tag{B.14}$$

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